

# Estimating the Grains of Sand on the Beaches of the World

Bruce Emerson

January 22, 2017

## 1 Why might I care?

The Grains of Sand estimation exercise hopes to address a variety of ideas. One is just working on our core math skills using exponents and scientific notation along with the concept of density from our previous lives. Another idea is to bring the value of dimensional analysis to the fore as a central tool in science and engineering. Yet another is my ongoing effort to help students develop reference points for the scale of various numbers that we encounter like Avogadro's number and the idea of a billion of anything. The specific scale we are seeking to get a better grip on here is how does Avogadro's number (the number of atoms in a mole) compare to number of grains of sand on all the beaches of the whole world. Just the beaches – not the ocean floor or the continents — just the beaches. This also allows us to explore the importance of being clear about our assumptions so that we may have fruitful conversations with others who would make different assumption.

## 2 Getting Started: Dimensional Analysis

There are basically two factors we need to figure out as most people see it. How much beach is there in the world and how much sand is there in a chunk of beach? A reasonable place to start is to assess the relationships between these factors (variables) and the quantity we are seeking. One could express the relationships this way:

more beach  $\implies$  more grains  
and  
more grains of sand in a 'chunk'  $\implies$  more grains overall

This means that each of these factors is what we call directly proportional. As a result we can say with confidence that, except for a scaling constant, our calculation will look like this:

grains of sand  $\approx$  (scaling factor)  $\cdot$  (how much beach)  $\cdot$  (grains of sand in a chunk)

Let's allow the scaling factor to be represented by  $C$  (a constant) and consider the units (dimensions) of the other factors. It does not seem like a big stretch to suggest that we might consider a cubic meter ( $m^3$ ) as the unit of a chunk of beach. This also suggests that we might describe the amount of beaches in the world in the same units. Let's look at just the dimensions in this context. To do this we replace each factor in the previous expression with the expected units and see if the left and right side of the equation make sense.

$$\text{grains} = (\text{no units}) \cdot (m^3) \cdot \left(\frac{\text{grains}}{m^3}\right)$$

We have grains of sand on the left and, after cancelling the  $m^3$ , we have grains of sand on the right. Since both the volume of all the beaches and the density of grains/ $m^3$  need no scaling factor we can set  $C=1$  without disrupting our estimation.

### 3 Size of the Beach

To determine the size of the 'World Beach' we are essentially calculating a volume which is length  $\cdot$  width  $\cdot$  depth. This will give us a volume in  $m^3$  which is consistent with our previous dimensional analysis. Let's take each term in turn.

#### 3.1 Width of Beach

One skill that we are trying to develop in physics is the ability to set boundaries on problems. It often helps to define what is too small to be reasonable and too big to be reasonable. It feels safe, to me, to assert that it's not really a beach if it's only 1m wide and I have very rarely been on a beach that is 100 m wide (a football/soccer field) even at low tide. In an order of magnitude sense taking a typical beach to be 10 m wide is the middle of this range.

$$\text{width: } 1m \ll 10m \ll 100m$$

#### 3.2 Depth of Beach

Similarly I would put the depth of a typical beach, based on personal excavating experience, at more than 0.1 m and definitely less than 10 m. Somewhere between 1 m and 2 m feels reasonable.

$$\text{depth: } 0.1m \ll 1m \ll 10m$$

#### 3.3 Length of Beach

Figuring out how long the collection of beaches of the world are is a much more challenging question and one that we could certainly have a very rich discussion about. Like most estimation problems I'm going to start by taking a swing

at it and then see what would happen if my estimated length were to change dramatically.

There are a couple of points to raise initially. How much of a coastline should we assume is beach as opposed to rocky and what is the impact of all the wiggles and inlets on this discussion. Oregon's beaches are probably not totally representative but it's hard to say. Even Hawai'i isn't beach everywhere so it's probably not unreasonable to estimate that half of the coastline is beach like though it will be more in some areas and definitely less in others. All the wiggles in the coastline seems like it would typically double, triple, or quadruple the actual length of the coastline. If I wanted to look into this more deeply I would consider finding studies about the fractal nature of coastlines. Here's how I might express this all mathematically.

0.5· linear length of coast is beach  
 increase by factor of 2 or 3 to account for wiggles  
 $\implies$  beach length  $\approx$  simple map length

Looking at a world map it seems like the coast of the western hemisphere (down the west coast from Alaska and up the east coast from Tierra del Fuego) looks like about once around the world. Not sure what to do about the islands yet but we'll address that in a minute. Using this same VERY rough approach I would suggest that Europe and Africa are another circumference and Asia together with Australia are yet another. Taking all the islands together might make another circumference though I personally doubt it but let's be charitable. Collectively that is about 4 circumferences of the world.

So what is the circumference of the world? A good number to know is the radius of the earth that we live on. It comes up in a number of contexts and is 6400 km to two significant figures. This yields a circumference of:

$$\begin{aligned} \text{Circumference of Earth } (C_{earth}) &= 2 \cdot \pi \cdot r \\ 40 \cdot 10^3 km &= 2 \cdot \pi \cdot 6.4 \cdot 10^3 km \\ C_{earth} &= 4 \cdot 10^7 m \end{aligned}$$

### 3.4 Putting it all together

Putting all these numbers together we get:

$$\begin{aligned} \text{Volume of Beachs} &= 10m \cdot 1m \cdot 4 \cdot (4 \cdot 10^7 m) \\ \text{Volume of Beachs} &= 1.6 \cdot 10^8 m^3 \text{ lower bound} = 1.6 \cdot 10^5 m^3 \\ \text{upper bound} &= 1.6 \cdot 10^{10} m^3 \end{aligned}$$

$$\text{Volume of Beachs} \approx 2 \cdot 10^8 m^3$$

## 4 Density of Sand Grains

So how big are sand grains? A reasonable starting point is to picture how many sand grains would fit in a mm (we're going to work entirely in the SI system of

units because it's what all the rest of the world uses so we all need to be able to work in the global economy). In my experience it seems like it would be pretty coarse sand if I only got 3 grains in a mm and it would be very fine sand if there were 10 grains in a mm. 6 grains/mm seems like a comfortable number which means 6000 grains in a meter. Now I need to assemble a cubic meter of sand which is 6000 grains x 6000 grains x 6000 grains....

$$\text{Density of Sand}/m^3 = (6 \cdot 10^3)^3$$

$$\text{Density of Sand}/m^3 = 216 \cdot 10^9$$

$$\text{Density of Sand}/m^3 \approx 2 \cdot 10^{11} \frac{\text{grains}}{m^3}$$

## 5 Total Grains

Having sorted out what we think a reasonable estimate of the factors are now we can proceed with the ultimate target.

$$\begin{aligned} \text{grains} &= (\text{no units}) \cdot (m^3) \cdot \left(\frac{\text{grains}}{m^3}\right) \\ \text{grains} &\approx 1.6 \cdot 10^8 m^3 \cdot 2 \cdot 10^{11} \frac{\text{grains}}{m^3} \end{aligned}$$

$$\text{grains} \approx 4 \cdot 10^{19}$$

Assuming the absolute worst case my numbers could be off by a factor of 1000 either way although statistically it's very unlikely that I would be off in the same direction by the maximum amount every time. We can express that worst case as follows:

$$4 \cdot 10^{16} \ll \text{grains} \approx 4 \cdot 10^{19} \ll 4 \cdot 10^{22}$$

Because, I hope, I've been very clear about where my numbers come from we can have an honest discussion about this calculation if we choose to. That's an important idea I hope you notice.

## 6 Meaning

We started this discussion in class by asking whether you imagined that the number of grains of sand on all the beaches of the world was a number bigger than, smaller than, or comparable to Avogadro's number. The class was divided roughly equally between those who thought it would be comparable and those who thought the grains of sand were greater than Avogadro's number. Since Avogadro's number is  $6.02 \cdot 10^{23}$  I hope this gives you a slightly different sense of how big that number is. One way of expressing this is to say that the number of atoms in your little finger is similar to all the grains of sand on all the beaches of 10,000 earths. Hopefully that makes you stop and think a little.....