

Project: Rotational Mathematics

In this project, we'll focus on applications of circles. I've noticed that many students dislike these kinds of problems, when, in all actuality, they're very straightforward and governed by only a few rules. Also, our text really doesn't get into them enough; when you take later math and physics classes, you'll need a good working knowledge of these ideas.

Before we begin, it'll behoove us to recall a couple of ideas from class. The **angular speed** (ω) of a rotating object is defined to be the number of radians (θ) that a central angle of a circle per unit time (t)...in formula form, $\omega = \frac{\theta}{t}$. The **linear speed** (v) of a rotating object is the arc length traveled by a point on the object (s) per unit time (t)...in formula form, $v = \frac{s}{t}$.

Two neat facts can be discerned from these ideas:

- 1) If an object spins about an axis, all points on the object are spinning with the same angular speed.
- 2) If an object is spinning, the farther an object is from the center of rotation, the faster its linear speed.

To convince ourselves of the truth of these statements, imagine climbing aboard the high wheel bike at right¹. Once you're safely aboard, imagine turning the pedal cranks one rotation (2π or 360°). The wheel is attached to the pedals rigidly; this means there is no freewheel. In other words, if you turn the pedal cranks one rotation, the hub that the pedal cranks are attached to also turns one rotation, as do each spoke on the wheel, as does the rim, as does the tire...you get the idea? And, they all make this rotation in the same amount of time. Thus, point 1) above makes sense.

Now for point 2)...clearly, even though each element of the wheel is turning at the same angular speed, the further away from the center of rotation you get, the more distance must be covered in a rotation, right? I mean, after all, some point at a distance of r from the center of rotation has to travel $2\pi r$ in one spin. So, it's clear that linear velocity increases as the radius increases.



Example 1: Suppose the high wheel bike shown above has a large wheel diameter of 52". Also, suppose its rear wheel has a diameter of 14". Suppose we climb aboard and turn the pedal cranks 20 times, keeping both wheels on the ground, in 20 seconds.

- a) How far forward does the bike travel?
- b) How fast are we traveling, on average?
- c) How many times does the rear wheel turn?

¹ Oh, how I wish I could.

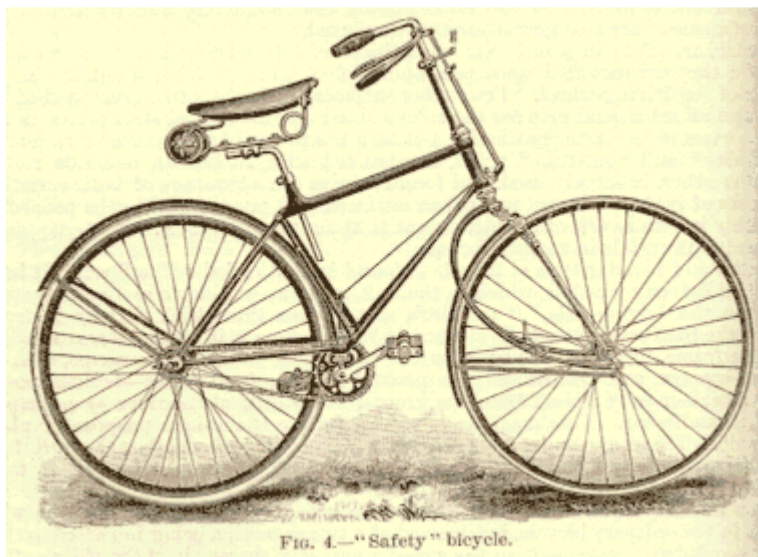
Solution 1a): Since the pedals turn 20 times, then the front tire makes 20 revolutions. We know that, with each rotation of the front wheel, this bike covers 52π inches on the ground (why?). So, after 20 rotations, we have covered $(20)(52\pi)$ inches, which is just about 272 feet!

Solution 1b): We have traveled this distance in 20 seconds, so our linear velocity is $\frac{(20)(52\pi \text{ inches})}{20 \text{ seconds}}$, or about 13.6 feet per second (just over 9 miles per hour) on average.

Solution 1c): Now, things change a tad. But, one thing stays the same: the fact the front wheel of this bicycle traveled $(20)(52\pi)$ inches in 20 seconds. If the front wheel covered this distance, the back wheel must have, as well. However, since the rear wheel has a smaller diameter than the front wheel, it must rotate more in order to cover this distance in the same period of time. How many rotations?

$$\frac{\text{distance traveled}}{\text{circumference of rear wheel}} = \frac{(20)(52\pi) \text{ inches}}{14\pi \text{ inches}} \approx 74 \text{ times}$$

As a self check, we could calculate the linear velocity of both the front and rear wheels to make sure that they are, indeed, spinning at the same rate. I encourage you to do just that!



The above high wheeled bicycle was a popular design for awhile...until folks started noticing that, when they hit something, they went careening over the handlebars almost instantaneously. The new design of bicycle that developed, thus, was called the “safety bicycle”. There’s one on the left.

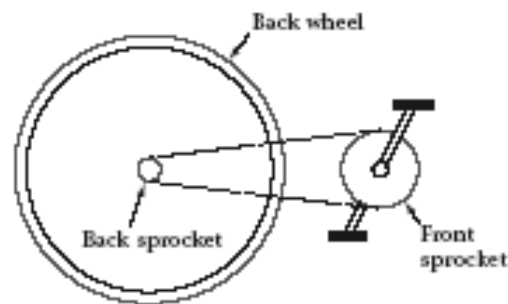
The bikes you and I ride on today are basically safety bikes, in design...and they lead us right into our next example (and connection to the last).

Example 2: On modern bicycles, a chain connects a front sprocket to a back sprocket. This is to force the back sprocket to turn with the same linear speed as the front. Suppose, in this configuration, that a cyclist is pedaling at 80 rpm (revolutions per

minute). Suppose, further, that the front sprocket has 53 teeth (all equal in size) while the back sprocket has 14 teeth (again, all equal in size, and equal in size to the front sprocket teeth). If the back wheel of this bicycle is 27.5” in diameter, how fast is the cyclist moving? Assume no coasting.

Solution 2: Well, I didn’t supply you with radii of the two sprockets, did I? No matter; knowing the number teeth is good enough². We have to notice, at first, that three things are spinning:

the front sprocket, the back sprocket, and the back wheel³. First, notice that the back wheel and back sprocket are turning at the same angular speed, since they are connected with an axle. Then, notice that



² In fact, in discussing gear ratios, we bike geeks never talk about diameters or radii of gears. We simply talk about teeth counts. We’re *that* cool.

³ The pedals are also spinning, and at their own rate, but since the pedals are bonded to the front sprocket, I see no need to discuss them. When you take physics, you’ll discuss torque, which is a function (partially) of the radius; thus, the longer the crank arms, the greater the torque, and the more power you can potentially create with each pedal stroke. But I’ll let Bruce tell you all about that.

the chain forces the front sprocket to turn the same linear speed as the back sprocket. Why? Well, since the sprockets are connected by the chain, as one gear rotates through a certain distance, the other gear has to match that distance in the same period of time, meaning that each gear travels the same distance per unit time. Let's utilize these two ideas in this problem.

We know the cyclist is pedaling at 80 rpm. How many rpm is the rear wheel making? Since the front sprocket has 53 teeth in its circumference, and the rear has 14, for every one rotation of the front sprocket, the back must rotate $\frac{53}{14}$ times (convince yourself why this must be true). Thus, if the pedals are making 80 rpm, the back sprocket is making $80 * \frac{53}{14}$, or almost 303 rpm.

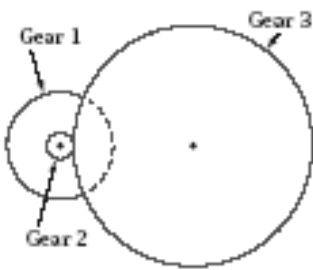
The back wheel, since it's bound with an axle to the back sprocket, must also be making the same rpm. However, we are interested in the speed of the bicycle, so we must turn rpm into linear speed. To do this, we can exploit a neat little relationship between angular speed and linear speed.

Starting with linear speed, let's apply the formula for arc length (from class) and see what happens:

$$v = \frac{s}{t} = \frac{r\theta}{t} = r \frac{\theta}{t} = r\omega$$

A-ha! So, if we have the angular velocity of an object, multiplying by the radius gives the linear velocity! Since the rear wheel is turning at roughly 303 rpm, this means its angular speed is roughly 1903 radians per minute (why?). With a diameter of 27.5", the wheel has a radius of 13.75", so the linear velocity of the rear wheel (and, thus, the cyclist) is $13.75" * \frac{1903 \text{ radians}}{\text{minute}}$, or about 26,165 inches per minute. Using a more sensible set of units, this comes to about 25 miles per hour.

Of course, we don't need to focus exclusively on bicycles...these same ideas translate perfectly to other applications. Let's look at gearing:



Example 3: In the diagram at left is a set of three gears (without their teeth, for clarity) designed to “gear down” a system. Gear 1 (of radius 8 inches) is bolted to gear 2 (of radius 2 inches), which drives gear 3 (of radius 20 inches). If gear 1 rotates at 300 rpm, how fast is gear 3 turning?

Solution 3: Do you see how this is similar to the previous problem? There's no longer a chain, but there might as well be between gears 2 and 3. Gears 1 and 2 are spinning at the same rate (300 rpm). However, their linear velocities differ because of their different radii; gear 2's linear speed is less (hence, the first part of helping to “gear down” the motor). Let's begin by finding gear 2's linear speed:

$$v = \frac{300 \text{ revolutions}}{\text{minute}} * 2\pi(2 \text{ inches}) \approx 3770 \text{ inches per minute}$$

Since gear 3 is driven by gear 2, it must be traveling at the same linear speed. However, due to its larger radius, it doesn't need to make as many rotations to accomplish this speed. How many does it have to make in a minute?

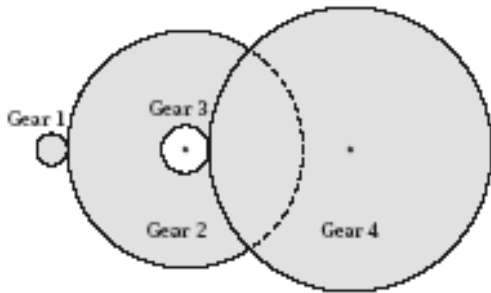
$$\# \text{ of rotations gear 3 must make} = \frac{3770 \text{ inches per minute}}{2\pi(20 \text{ inches})} = 30 \text{ rpm}$$

So we say that gear 2 and gear 3 are in a 1 to 10 gear ratio, since the rpm of gear 3 is ten times less than the rpm of gear 2 (and gear 1). Hence, the system has been geared down (say, for increasing torque).

Well, I think we've done enough so that you all get the big ideas. Circles bolted together have the same angular velocity. Circles joined by chains (or driving each other directly) have the same linear velocity. That's really all you need; run with it!

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(2 points) (w) My wife's favorite bike is her 1963 Schwinn Fleet. It has a beefy drive train: the front sprocket has 50 teeth, while the back sprocket only has 18. Additionally, the rear wheel is 28" in diameter. If she's pedaling down the road at 60 rpm, how fast is she going in miles per hour, to the nearest mile per hour?



Ever watch your car's tachometer when you're driving⁴? Your engine rotates pretty quickly! It actually spins much more quickly than, say, the tires it's driving. In order to accomplish this safely, engineers use a **gear train**. There's a picture of one at left, say, for a car's 4WD low gearing. In this gear train, gear 1 is rotating at 2700 rpm. Gear 1's teeth drive gear 2, which is bolted to gear 3. Gear 3's teeth, in turn, drive gear 4. Suppose the gears have the following radii:

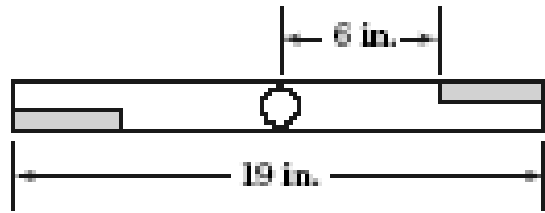
Gear	1	2	3	4
Radius (cm)	2	15	3	18

- a) **(4 points) (w)** How many rpm is gear 4 making?
- b) **(3 points) (w)** Suppose that a wheel and tire (with combined diameter of 28") are connected to gear 4 with an axle. How fast is this car traveling in miles per hour (still assuming gear 1 is spinning at 2700 rpm) to the nearest mile per hour?

(4 points) (w) True story...a student of mine (let's call him Rufus) was pulled over for speeding recently. The police officer told him he was going 5 over the posted speed limit of 35 miles per hour, but Rufus swore his speedometer indicated exactly 35 mph at the moment he was "radared" by the police. Later, he remembered that he had placed a new set of tires on his car the previous week. His car (and thus, his speedometer) was designed for 26.5" inch diameter tires, but he had replaced all four tires with new 29" diameter tires. If, indeed, Rufus' speedometer said he was traveling at 35 mph when he was pulled over by the police officer, how fast was he *really* traveling, to the nearest mile per hour?

⁴ Well, not *too* much. You should be watching the road, I suppose..

The blade on a rotary lawnmower is 19 inches long. The cutting edges begin 6 inches from the center of the blade. In order for the blade to cut grass effectively, the innermost part of the cutting edge must strike the grass at a speed of 900 inches per second.



- a) **(4 points) (w)** How many rpm must the blade make in order for this to occur, to the nearest rpm?
- b) **(3 points)** Assume that the blade is turning at the number of rpm from part a). Now suppose, as you're cutting your lawn with this lawnmower, you strike a stone at the outermost tip of the blade. How fast will the stone be ejected away from the blade, to the nearest mile per hour?

(extra 2 points) A spinning figure skater brings her arms closer into her body. At that moment, she begins spinning faster. Why? Use *only* ideas from this project, please!