8.3 Expected Value

- Find an expected value involving two events.
- Find an expected value involving multiple events.
- Use expected value to make investment decisions.

Finding an Expected Value Involving Two Events

The **expected value** of an "experiment" is the long-run average—if the experiment could be repeated many times, the expected value is the average of all the results.

Expected Value

Consider an experiment that has only two possible events. The expected value of the experiment is

$$\text{Expected value} = \begin{pmatrix}\text{probability}\\ \text{of event 1}\end{pmatrix}\begin{pmatrix}\text{payoff for}\\ \text{event 1}\end{pmatrix} + \begin{pmatrix}\text{probability}\\ \text{of event 2}\end{pmatrix}\begin{pmatrix}\text{payoff for}\\ \text{event 2}\end{pmatrix}.$$

EXAMPLE 1 Finding an Expected Value

In a state lottery, a single digit is drawn from each of four containers. Each container has 10 balls numbered 0 through 9. To play, you choose a 4-digit number and pay \$1. If your number is drawn, you win \$5000. If your number is not drawn, you lose your dollar. What is the expected value for this game?

SOLUTION

The probability of winning is 1/10,000. The probability of losing is 9999/10,000.

$$\text{Expected value} = \left(\frac{1}{10{,}000}\right)(4999) + \left(\frac{9999}{10{,}000}\right)(-1) = -\$0.50$$

Probability of win · Payoff for win · Probability of lose · Payoff for lose

So, on average, you should expect to lose \$0.50 each time you play the game.

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Play the *Lottery Simulator* at *Math.andYou.com*. Set the number of games to 100,000. Discuss your results in the context of expected value. States are often criticized for falsely raising people's expectations of winning and for encouraging a form of regressive tax on the poor. What is your opinion of this?

The total annual cost of fire in the United States is about 2.5% of the domestic gross product.

EXAMPLE 2 Find an Expected Value

You take out a fire insurance policy on your home. The annual premium is $300. In case of fire, the insurance company will pay you $200,000. The probability of a house fire in your area is 0.0002.

a. What is the expected value?

b. What is the insurance company's expected value?

c. Suppose the insurance company sells 100,000 of these policies. What can the company expect to earn?

SOLUTION

a. Expected value $= (0.0002)(199{,}700) + (0.9998)(-300) = -\$260.00$

$200{,}000 - 300$ (Fire) (No Fire)

The expected value over many years is $-\$260$ per year. Of course, your hope is that you will never have to collect on fire insurance for your home.

b. The expected value for the insurance company is the same, except the perspective is switched. Instead of $-\$260$ per year, it is $+\$260$ per year. Of this, the company must pay a large percent for salaries and overhead.

c. The insurance company can expect to gross $30,000,000 in premiums on 100,000 such policies. With a probability of 0.0002 for fire, the company can expect to pay on about 20 fires. This leaves a gross profit of $26,000,000.

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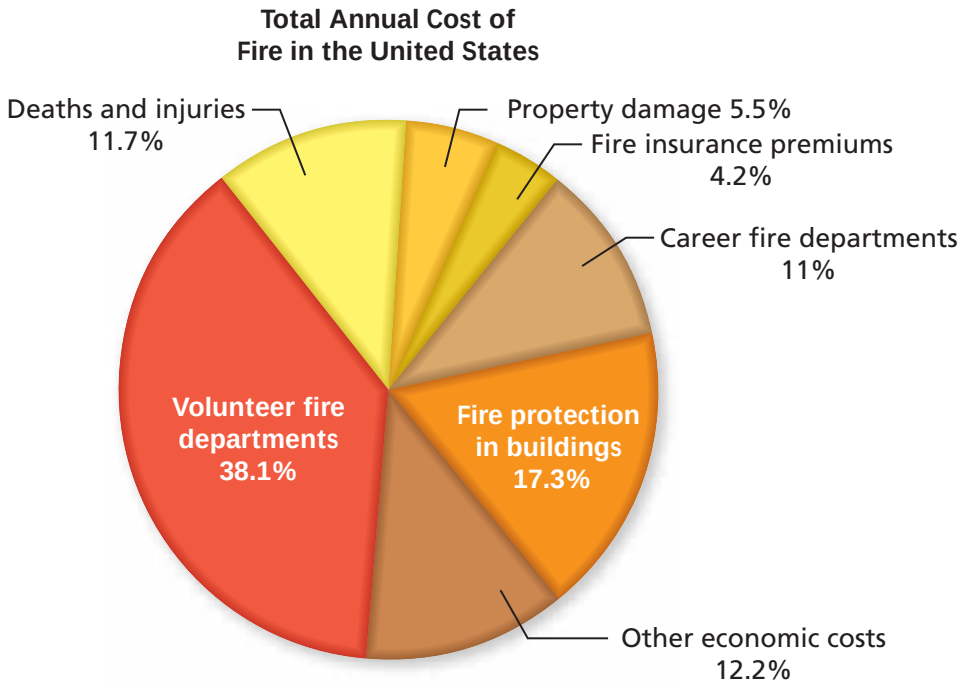
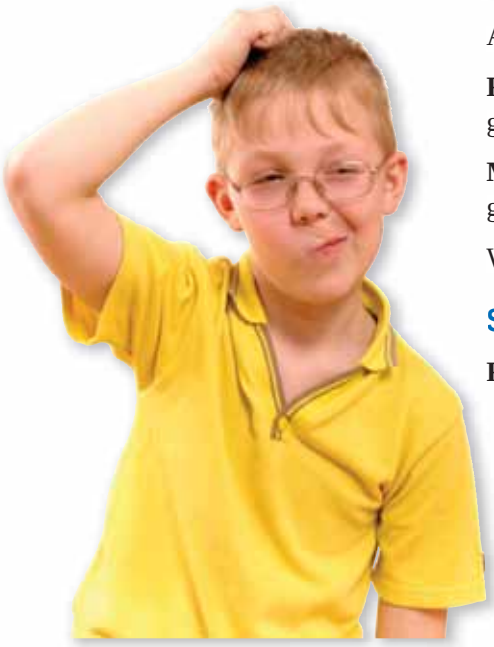
In the circle graph, why is the percent for property damage greater than the percent for fire insurance premiums?



Finding an Expected Value Involving Multiple Events

EXAMPLE 3 Comparing Two Expected Values



A child asks his parents for some money. The parents make the following offers.

Father's offer: The child flips a coin. If the coin lands heads up, the father will give the child $20. If the coin lands tails up, the father will give the child nothing.

Mother's offer: The child rolls a 6-sided die. The mother will give the child $3 for each dot on the up side of the die.

Which offer has the greater expected value?

SOLUTION

Father's offer:

$$\text{Expected value} = \left(\frac{1}{2}\right)(20) + \left(\frac{1}{2}\right)(0) = \$10$$

Probability of heads — Payoff for heads — Probability of tails — Payoff for tails

Mother's offer: There are six possible outcomes.

DATA

	A	B	C	D
1	**Number**	**Payoff**	**Probability**	**Expected Value**
2	1	$3.00	16.67%	$0.50
3	2	$6.00	16.67%	$1.00
4	3	$9.00	16.67%	$1.50
5	4	$12.00	16.67%	$2.00
6	5	$15.00	16.67%	$2.50
7	6	$18.00	16.67%	$3.00
8	**Total**			**$10.50**

Even though the mother's offer has a slightly higher expected value, the best the child can do with the mother's offer is $18, whereas the child has a 50% chance of receiving $20 with the father's offer.

Checkpoint

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The child's uncle makes a different offer. The child rolls a 12-sided die. The uncle will give the child $2 for each dot on the up side of the die. Use a spreadsheet to find the expected value of this offer. Which offer would you take? Explain.

	A	B	C	D
1	**Number**	**Payoff**	**Probability**	**Expected Value**
2	1	$2.00	8.33%	$0.17
3	2	$4.00	8.33%	$0.33
4	3	$6.00	8.33%	$0.50
5	4	$8.00	8.33%	$0.67

EXAMPLE 4 Using a Decision Tree

Your company is considering developing one of two cell phones. Your development and market research teams provide you with the following projections.

As of 2010, it was estimated that there were over 5 billion cell phone subscriptions worldwide. With this massive market, the enticement to invest in the development of new and innovative products is strong.

Cell phone A:

Cost of development: $2,500,000

Projected sales: 50% chance of net sales of $5,000,000
30% chance of net sales of $3,000,000
20% chance of net sales of $1,500,000

Cell phone B:

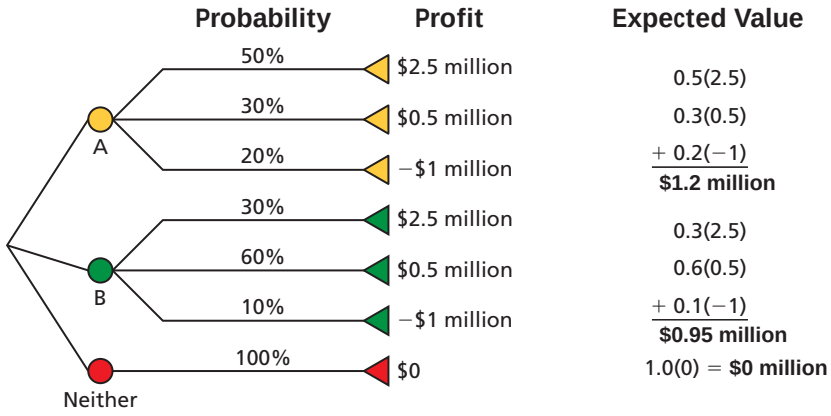
Cost of development: $1,500,000

Projected sales: 30% chance of net sales of $4,000,000
60% chance of net sales of $2,000,000
10% chance of net sales of $500,000

Which model should your company develop? Explain.

SOLUTION

A *decision tree* can help organize your thinking.



Although cell phone A has twice the risk of losing $1 million, it has the greater expected value. So, using expected value as a decision guideline, your company should develop cell phone A.

Checkpoint

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Which of the following should your company develop? Explain.

Cell phone C: Cost of development: $2,000,000

Projected sales: 40% chance of net sales of $5,000,000
40% chance of net sales of $3,000,000
20% chance of net sales of $1,500,000

Cell phone D: Cost of development: $1,500,000

Projected sales: 15% chance of net sales of $4,000,000
75% chance of net sales of $2,000,000
10% chance of net sales of $500,000

Using Expected Value to Make Investment Decisions

Daniel Kahneman, a professor at Princeton University, became the first psychologist to win the Nobel Prize in Economic Sciences. The prize was awarded for his "prospect theory" about investors' "illusion of control."

EXAMPLE 5 Using Expected Value

Analyze the mathematics in the following description of Daniel Kahneman and Amos Tversky's "Prospect Theory: An Analysis of Decision under Risk."

> "A problem is positively framed when the options at hand generally have a perceived probability to result in a positive outcome. Negative framing occurs when the perceived probability weighs over into a negative outcome scenario. In one of Kahneman and Tversky's (1979) experiments, the participants were to choose one of two scenarios, an 80% possibility to win \$4,000 and the 20% risk of not winning anything as opposed to a 100% possibility of winning \$3,000. Although the riskier choice had a higher expected value ($\$4{,}000 \times 0.8 = \$3{,}200$), 80% of the participants chose the safe \$3,000. When participants had to choose between an 80% possibility to lose \$4,000 and the 20% risk of not losing anything as one scenario, and a 100% possibility of losing \$3,000 as the other scenario, 92% of the participants picked the gambling scenario. This framing effect, as described in . . . Prospect Theory, occurs because individuals over-weigh losses when they are described as definitive, as opposed to situations where they are described as possible. This is done even though a rational economical evaluation of the two situations lead to identical expected value. People tend to fear losses more than they value gains. A \$1 loss is more painful than the pleasure of a \$1 gain." Johan Ginyard

SOLUTION

Here are the first two options the participants were given.

Greater expected value

		Expected Value
Option 1:	80% chance of gaining \$4000 20% chance of gaining \$0	$(0.8)(4000) + (0.2)(0) = \$3200$
Option 2:	100% chance of gaining \$3000	$(1.0)(3000) = \$3000$

Preferred by participants

Here are the second two options the participants were given.

Preferred by participants

		Expected Value
Option 1:	80% chance of losing \$4000 20% chance of losing \$0	$(0.8)(-4000) + (0.2)(0) = -\$3200$
Option 2:	100% chance of losing \$3000	$(1.0)(-3000) = -\$3000$

Greater expected value

What Kahneman and Tversky found surprising was that in neither case did the participants intuitively choose the option with the greater expected value.

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Describe other situations in which people fear losses more than they value gains.

From 1973 through 2010, the Standard and Poor 500 Index had an average annual gain of 11.5%. During this time, its greatest annual gain was 37.4% in 1995, and its greatest annual loss was 37% in 2008.

EXAMPLE 6 Comparing Two Expected Values

A *speculative investment* is one in which there is a high risk of loss. What is the expected value for each of the following for a $1000 investment?

a. Speculative investment

- Complete loss: 40% chance
- No gain or loss: 15% chance
- 100% gain: 15% chance
- 400% gain: 15% chance
- 900% gain: 15% chance

b. Conservative investment

- Complete loss: 1% chance
- No gain or loss: 35% chance
- 10% gain: 59% chance
- 20% gain: 5% chance

SOLUTION

a. Speculative investment

DATA

	A	B	C	D
1	**Result**	**Payoff**	**Probability**	**Expected Value**
2	Complete loss	-$1,000	40%	-$400
3	No gain or loss	$0	15%	$0
4	100% gain	$1,000	15%	$150
5	400% gain	$4,000	15%	$600
6	900% gain	$9,000	15%	$1,350
7	**Total**		**100%**	**$1,700**

b. Conservative investment

	A	B	C	D
1	**Result**	**Payoff**	**Probability**	**Expected Value**
2	Complete loss	-$1,000	1%	-$10
3	No gain or loss	$0	35%	$0
4	10% gain	$100	59%	$59
5	20% gain	$200	5%	$10
6	**Total**		**100%**	**$59**

This example points out the potential gain and the risk of investment. The speculative investment has an expected value of $1700, which is a high return on investment. If you had the opportunity to make 100 such investments, you would have a high likelihood of making a profit. But, when making only 1 such investment, you have a 40% chance of losing everything.

Checkpoint

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Which of the following investments is better? Explain your reasoning.

c. Speculative investment

- Complete loss: 20% chance
- No gain or loss: 35% chance
- 100% gain: 35% chance
- 400% gain: 5% chance
- 2000% gain: 5% chance

d. Conservative investment

- Complete loss: 2% chance
- No gain or loss: 38% chance
- 20% gain: 55% chance
- 30% gain: 5% chance

8.3 Exercises

Life Insurance **The table shows the probabilities of dying during the year for various ages. In Exercises 1–6, use the table.** ***(See Examples 1 and 2.)***

Probability of Dying During the Year		
Age	**Male**	**Female**
21	0.001420	0.000472
22	0.001488	0.000487
23	0.001502	0.000496
24	0.001474	0.000503
25	0.001430	0.000509
26	0.001393	0.000519
27	0.001366	0.000535
28	0.001362	0.000561
29	0.001379	0.000595
30	0.001406	0.000637

1. A 23-year-old male pays $275 for a 1-year $150,000 life insurance policy. What is the expected value of the policy for the policyholder?

2. A 28-year-old female pays $163 for a 1-year $200,000 life insurance policy. What is the expected value of the policy for the policyholder?

3. A 27-year-old male pays $310 for a 1-year $175,000 life insurance policy. What is the expected value of the policy for the insurance company?

4. A 25-year-old female pays $128 for a 1-year $120,000 life insurance policy. What is the expected value of the policy for the insurance company?

5. A 26-year-old male pays $351 for a 1-year $180,000 life insurance policy.

a. What is the expected value of the policy for the policyholder?

b. What is the expected value of the policy for the insurance company?

c. Suppose the insurance company sells 10,000 of these policies. What is the expected value of the policies for the insurance company?

6. A 30-year-old female pays $259 for a 1-year $250,000 life insurance policy.

a. What is the expected value of the policy for the policyholder?

b. What is the expected value of the policy for the insurance company?

c. Suppose the insurance company sells 10,000 of these policies. What is the expected value of the policies for the insurance company?

Consumer Electronics Company **A consumer electronics company is considering developing one of two products. In Exercises 7–10, use a decision tree to decide which model the company should develop.** ***(See Examples 3 and 4.)***

7. Laptop A: Cost of development: $8 million

Projected Sales	
Probability	**Net sales (in millions)**
10%	$16
70%	$12
20%	$6

Laptop B: Cost of development: $10 million

Projected Sales	
Probability	**Net sales (in millions)**
30%	$18
50%	$14
20%	$8

8. MP3 Player A:
Cost of development: $5 million

Projected Sales	
Probability	**Net sales (in millions)**
20%	$10
60%	$8
20%	$2

MP3 Player B:
Cost of development: $3 million

Projected Sales	
Probability	**Net sales (in millions)**
40%	$6
40%	$5
20%	$1

9. E-reader A: Cost of development: $3 million

Projected Sales	
Probability	**Net sales (in millions)**
20%	$10
45%	$6
25%	$5
10%	$0.5

E-reader B: Cost of development: $4 million

Projected Sales	
Probability	**Net sales (in millions)**
10%	$12
40%	$10
30%	$4
20%	$1

10. Camera A:
Cost of development: $5 million

Projected Sales	
Probability	**Net sales (in millions)**
10%	$13
30%	$10
20%	$8
25%	$6
15%	$2

Camera B:
Cost of development: $3.5 million

Projected Sales	
Probability	**Net sales (in millions)**
20%	$10
35%	$7.5
25%	$5.5
10%	$3.5
10%	$0.5

Option Comparison **In Exercises 11–14, compare the two options.** ***(See Example 5.)***

11.

	Probability	Gain
Option 1	100%	$1000
	0%	$0
Option 2	60%	$2000
	40%	$0

12.

	Probability	Gain
Option 1	100%	−$1000
	0%	$0
Option 2	70%	−$2000
	30%	$1000

13.

	Probability	Gain
Option 1	80%	$1000
	20%	$3000
Option 2	90%	$2000
	10%	$0

14.

	Probability	Gain
Option 1	75%	$500
	25%	$1500
Option 2	50%	−$500
	50%	$2000

Investment Comparison **You want to invest $1000. In Exercises 15 and 16, find the expected values for the two investments.** ***(See Example 6.)***

15.

Speculative investment

- Complete loss: 10% chance
- No gain or loss: 20% chance
- 150% gain: 40% chance
- 200% gain: 20% chance
- 700% gain: 10% chance

Conservative investment

- Complete loss: 1% chance
- No gain or loss: 39% chance
- 50% gain: 40% chance
- 100% gain: 20% chance

16.

Speculative investment

- Complete loss: 30% chance
- No gain or loss: 25% chance
- 100% gain: 20% chance
- 500% gain: 15% chance
- 1000% gain: 10% chance

Conservative investment

- Complete loss: 5% chance
- No gain or loss: 15% chance
- 30% gain: 60% chance
- 60% gain: 20% chance

Extending Concepts

Investment Portfolio **The table shows the rates of return of two stocks for different economic states. In Exercises 17 and 18, use the table.**

		Rate of return	
Economic State	Probability	Stock V	Stock W
Boom	20%	28%	−5%
Normal	65%	12%	7%
Recession	15%	−16%	23%

17. Compare the expected rates of return of the two stocks.

18. You invest 50% of your money in stock V and 50% of your money in stock W. What is the expected rate of return?

Investment Portfolio **The table shows the rates of return of three stocks for different economic states. In Exercises 19–22, find the expected rate of return for the portfolio.**

		Rate of return		
Economic state	Probability	Stock X	Stock Y	Stock Z
Boom	20%	16%	27%	3%
Normal	65%	9%	13%	7%
Recession	15%	−5%	−23%	21%

19.

Portfolio mix:
50% invested in stock X
20% invested in stock Y
30% invested in stock Z

20.

Portfolio mix:
70% invested in stock X
10% invested in stock Y
20% invested in stock Z

21.

Portfolio mix:
50% invested in stock X
25% invested in stock Y
25% invested in stock Z

22.

Portfolio mix:
60% invested in stock X
30% invested in stock Y
10% invested in stock Z

13. The probabilities are the same because no matter how the data is presented, the overall probability of saying "No" is the same.

15. Underweight: About 14.9%; unlikely
Normal weight: About 23.3%; unlikely
Overweight: About 49.0%; about equally likely to happen or not happen
Obese: About 59.1%; somewhat likely
Morbidly obese: About 62.1%; likely

17. About 52.2%

19. About 19.6%

21. 3.5

23. 25% for all 4 suits

25. The experimental and theoretical probabilities for spades and clubs are about the same. The experimental probability for hearts is greater than the theoretical probability. The experimental probability for diamonds is less than the theoretical probability.

27. About 53.8%

8.1–8.2 Quiz *(page 372)*

1.

Month	Fraction	Decimal	Percent
November	$\frac{11}{250}$	0.044	4.4%
December	$\frac{25}{74}$	0.338	33.8%
January	$\frac{81}{1000}$	0.081	8.1%
February	$\frac{301}{1000}$	0.301	30.1%
March	$\frac{5}{34}$	0.147	14.7%

November: very unlikely
December: somewhat unlikely
January: very unlikely
February: unlikely
March: very unlikely

3. Likely

5. Nearly impossible

7. 70%

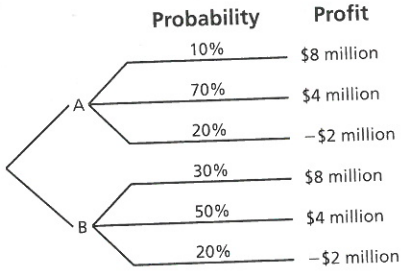
Section 8.3 *(page 380)*

1. −$49.70

3. $70.95

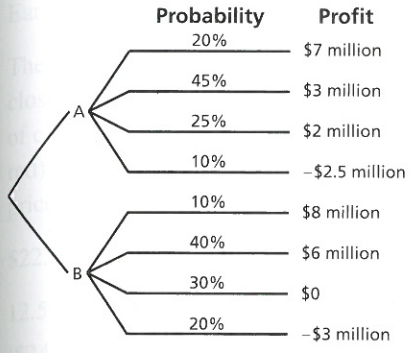
5. **a.** −$100.26 **b.** $100.26 **c.** $1,002,600

7.



The company should develop Laptop B.

9.



The company should develop E-reader A.

11. Expected value for Option 1: $1000
Expected value for Option 2: $1200
Option 2 has the greater expected value.

13. Expected value for Option 1: $1400
Expected value for Option 2: $1800
Option 2 has the greater expected value.

15. Expected value for speculative investment: $1600
Expected value for conservative investment: $390

17. Expected rate of return for Stock V: 11%
Expected rate of return for Stock W: 7%
Stock V has a greater expected rate of return.

19. 8.72%

21. 8.825%

Section 8.4 *(page 390)*

1. 0.1728%
3. 0.0512%
5. 0.0216%
7. 0.0008%
9. About 44.4%
11. About 98.4%
13. About 45.5%
15. About 77.0%
17. About 33.3%
19. If you do not switch, you win 2 out of 5 times. If you do switch, you win 3 out of 5 times. So, based on probability, you should switch.
21. 20%
23. About 2.0%
25. About 74.7%

8.3–8.4 Quiz *(page 394)*

1. 2.5 points
3. About 5.6%
5. About 1.9%
7. About 94.4%

Chapter 8 Review *(page 396)*

1.

Event	Fraction	Decimal	Percent
Being an organ donor	$\frac{4}{15}$	0.267	26.7%
Eats breakfast	$\frac{61}{100}$	0.61	61%
Having a dream that comes true	$\frac{429}{1000}$	0.429	42.9%
Household with television	$\frac{491}{500}$	0.982	98.2%

The probability of being an organ donor is unlikely.

The probability of eating breakfast is about equally likely to happen or not happen.

The probability of having a dream that comes true is about equally likely to happen or not happen.

The probability of a household having a television is almost certain.