Exponential Modeling Exercises

Pages	Problems
318 - 321	1-4, 7-12, 15-17 (also, for these data, also give the model implied by the data…is it exponential? Quadratic? Something else?), 19 – 24 (in 21 and 22, find the mathematical error in the formula in the book), 27, 28

The data at the right shows the cooling temperatures of a freshly brewed cup of coffee after it is poured from the brewing pot into a serving cup.

Time (mins)	Temp (º F)
0	179.5
5	168.7
8	158.1
11	149.2
15	141.7
18	134.6
22	125.4
25	123.5
30	116.3
34	113.2
38	109.1
42	105.7
45	102.2
50	100.5

E1. Determine an exponential regression model equation to represent this data.

E1(**a**). Why does an exponential model make sense in this case?

E2. According to your model, at what percentage rate is the coffee cooling, on average per minute?

E3. When is the coffee at a temperature of 106 degrees?

E4. Is your previous answer an extrapolation or interpolation?

E5. What is the predicted temperature of the coffee after 1 hour?

E6. Is your previous answer an extrapolation or interpolation?

E7. In 1992, a woman sued McDonald's for serving coffee at a temperature of 180^o that caused her to be severely burned when the coffee spilled. An expert witness at the trial testified that liquids at 180^o will cause a full thickness burn to human skin in two to seven seconds. It was stated that had the coffee been served at 155^o , the liquid would have cooled and avoided the serious burns. The woman was awarded over 2.7 million dollars. Because of this famous case, many restaurants now serve coffee at a temperature around 155^o. How long should restaurants wait (after pouring the coffee from the pot) before serving coffee, to ensure that the coffee is not hotter than 155^o?

E8. If the temperature in the room is 76° F, what will happen to the temperature of the coffee, after being poured from the pot, over an extended period of time?

The sad data below shows the United States' Public Debt from 1910 to 2007.

Year	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000	2005	2007
US Public Debt (billions of dollars)	2.6	25.9	16.2	43	257.4	290.2	389.2	930.2	3233	5674	7933	9008

E9. Find a logistic model for the US Public Debt (in billions of dollars).

E10. Find an exponential model for the US Public Debt (in billions of dollars).

E11. Which one most closely approximates the most recent US public debt?

E12. Remember that the logistic model involves a constant called "***e***". Like π, ***e*** is a pretty important mathematical constant. Do a little research to find out what ***e*** is.

Answers.

E1. $\boldsymbol{y} = 171.46(0.988)^{\boldsymbol{x}}$ ($\boldsymbol{y}$ = temperature in degrees Fahrenheit, $\boldsymbol{x}$ = minutes after pour)

E1(**a**). This is pretty cool.

So, we can't analyze this the exact same way that we did in class (that is, looking at successive quotients), since the data that we're given is not sequential in the **x** – values (that is, we go "0, 5, 8, 11…" instead of "0,1,2,3,…"). However, we **can** fill in the blanks and look to see if a pattern exists.

Let's think about this: If the temperatures are decreasing exponentially, then there's a fixed percentage that's falling off per unit of time (we'll use a minute as a unit of time). Therefore, I can use the extended chart that I created below to model this:

- After 1 minute, the 179.5 degrees will have fallen to (some percentage less than 1)*179.5. To be compact, we'll call that ***p****179.5, where ***p*** is known to be a decimal less than 1.
- After 2 minutes, we need to take ***p*** of (***p***)179.5 (that is, we take the same percentage on the new base value). This will then be ***p*** ****p****179.5, or $\boldsymbol{p}^2$*179.5
- After 3 minutes, there have been 3 (applications" of the percentage to the temperature, so we're now at $\boldsymbol{p}^3$*179.5.
- So, extending this idea, after ***n*** minutes, we will have applied the percentage ***p*** to 179.5 ***n*** times. So, at the nth minute, the temperature will be ($\boldsymbol{p}^{\boldsymbol{n}}$)*179.5

Keep in mind, however, that we don't know what ***p*** actually IS. We're just saying that, if this data is, indeed, exponential, that ***p*** will be a constant. OK…time for the algebra!

OK…now take a look at that column I added at left. According to the pattern described above, if the data is exponential, each of those ***p***'s should be roughly equal. To see if that's the case, we'll solve all of the implied equations at left, and see what ***p*** equals. By "implied equations", we'll check to see if, for example, P^5*179.5 = 168.7 yields the same value of ***p*** as p^8*179.5 = 158.1, as does p^1*179.5 = 149.2, and so on, all the way down.

Time (mins)	Temp (º F)	Temp (as expression of *p*)
0	179.5	P^{0}*179.5
5	168.7	P^{5}*179.5
8	158.1	P^{8}*179.5
11	149.2	P^{11}*179.5
15	141.7	P^{15}*179.5
18	134.6	P^{18}*179.5
22	125.4	P^{22}*179.5
25	123.5	P^{25}*179.5
30	116.3	P^{30}*179.5
34	113.2	P^{34}*179.5
38	109.1	P^{38}*179.5
42	105.7	P^{42}*179.5
45	102.2	P^{45}*179.5
50	100.5	P^{50}*179.5

Time (mins)	Temp (º F)	Temp (as expression of *p*)	*p*
0	179.5	P^{0}*179.5	*(n/a)*
5	168.7	P^{5}*179.5	0.988
8	158.1	P^{8}*179.5	0.984
11	149.2	P^{11}*179.5	0.983
15	141.7	P^{15}*179.5	0.984
18	134.6	P^{18}*179.5	0.984
22	125.4	P^{22}*179.5	0.984
25	123.5	P^{25}*179.5	0.985
30	116.3	P^{30}*179.5	0.986
34	113.2	P^{34}*179.5	0.987
38	109.1	P^{38}*179.5	0.987
42	105.7	P^{42}*179.5	0.987
45	102.2	P^{45}*179.5	0.988
50	100.5	P^{50}*179.5	0.988

Well, check ***that*** out!

Since the percentages are (roughly and arguably) equal (to about 98%, same as the model you found in E1), it must be that the data is exponential.

Rad!

E2. Since 0.988 stands for 98.8%, the coffee is cooling at a rate of 1.2% per minute (or, said a different way, “after each minute, the coffee is only 98.8% of its temperature from a minute ago”).

E3. According to the model, at around 41 minutes (notice the error from the data. By “error” I mean “good error”, not “bad error”).

E4. Interpolation.

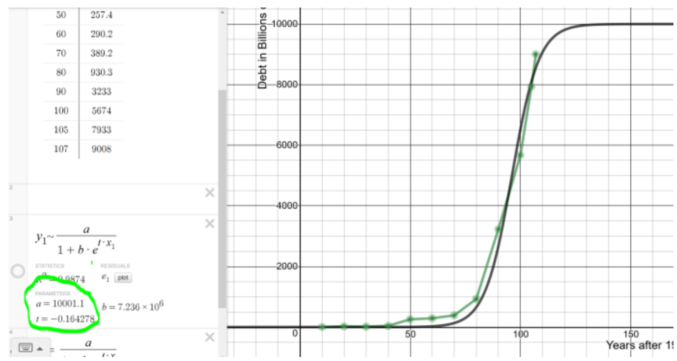
E5. About 84 degrees F.

E6. Extrapolation.

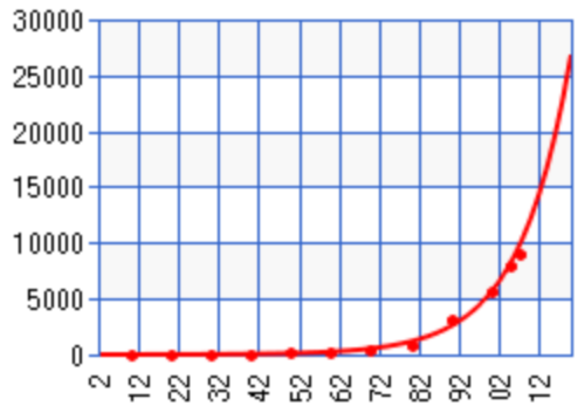
E7. The model predicts 8.5 minutes. The data says a little bit more than 8 minutes, so it agrees. Being a litigious society, I’d bet American lawyers would say 10 minutes, just to be safe. A better question is why this case was taken seriously (http://en.wikipedia.org/wiki/Liebeck_v._McDonald%27s_Restaurants)

E8. What do you think? Would the model curve look the same as the one you just found? If yes, why? If not, how would it change?

E9. Quiz 3 has a nice website you can check out for this one! At right is the model I crunched – logistics predict a leveling off, and this one predicts a leveling off at about $10000 billion (or $10 trillion). I guess we’ll see if that’s even remotely true.



E10. This time, I used https://keisan.casio.com/exec/system/14059930973581 and got the graph at right. The equation has a constant average multiplier of 1.08, which means that the predicted public debt is increaing by 8% each year. Doesn’t seem like much – until you realise that 8% of a trillion dollars is tens of ***billions*** of dollars. Said another way – in the time it took me to type this sentence, by this model, the US debt increased by about $10,00. #**sorry** [1]



E11. C’mon now! Go do a little Googling!

E12. A hint: ***e*** stands for “Euler”.

[1] Hell, that hashtag cost about $600 all by itself. (#**sorry**)[2]

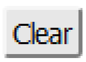
Quiz 1.

Some of you might keep chickens in town. You might find this interesting and/or relevant!

"***Your hen may live a long time, but she may not lay the entire time. By the time your hen reaches three years old, her egg output has dropped to 65 percent of her maximum laying ability. This declines…to where if your hen lives to 10 years old, she will only lay 20 percent of the maximum egg output of her first year***."

Source: http://animals.pawnation.com/life-expectancy-laying-hen-1727.html

Let's assume that, through her first year of life, a hen attains 100% of her laying capacity (this is pretty much true, at least based on my anecdotal evidence – they start laying around 6 months of age). Thus, this fact, plus the two facts from the paragraph above, give us three data points: **(0,1)**, **(3,0.65)**, and **(10,0.2)**.

1. (**2 points**) Use these three points to create an exponential regression model (it'll be in the form $y = AB^x$ form, where x is the number of years of life, and y is the percentage of laying capacity left). Here's a good site to help you! http://keisan.casio.com/exec/system/14059930973581 . To use it, press the 

button and then enter your data in the "**x**" and "**y**" boxes. Then, press 

. You'll have to assemble your equation by substituting in the "**A**" and "**B**" values from the table below the data…round the value of "**A**" to the nearest whole number and the value of "**B**" to two decimals.

Give me that equation!

2. (**4 points**) What percent of "maximum laying ability" does the average chicken ***lose*** each year (according to your model)?

3. (**2 points**) At what percent of maximum laying ability is a hen who lives to 8 years? Our oldest girl, Bertha, is that old right now! To get this value, you can either use the mathematics we did in class (that is, repeated multiplications), or use the graph that's supplied when you run the regression (or, if you're feelin' saucy, something else).[2]

4. (**2 points**) The Guinness Book of World Records has given the title of "World's Oldest Chicken" to a (now-deceased) hen named Matilda. She was 14 years old. What was her egg laying capacity at the end of her life? The graph runs out of space, but you can probably get pretty close by inspection (or run the math)

[2] You can interpret that number as "the percent of days these girls lay eggs". I'm gonna go ahead and say Bertha's WELL below that average. #**freeloader**.

Quiz 2.

One of the neatest connections of exponential growth can be applied to money – if you place your money into an interest – bearing account, you'll accrue said interest (as a "thank you" from the company who's borrowing your money), so you end up with more. We've talked about this in class! Here's an example: right now, Johnson & Johnson stock has a 3.6% interest payout. Assuming this percentage continues, here's what would happen to $1000 you invested with J&J stock:

Years Passed	Y = Value of Investment
0	$**1000**
1	$1000 (your initial investment) + 0.036*$1000 (the 3.6% added to your initial investment = $**1036**
2	$1036 (your new "initial investment" after year 1) + 0.036*$1036 = $**1073.30**
3	$1073.30 + 0.036*$1073.30 = $**1111.94**
4	$1111.94 + 0.036*$1111.94 = $**1151.96**

And so forth. Please note that you earn ***more*** interest each successive year, even though the interest rate is staying the same – that's just because the base amount (the part the interest is applied to) is getting bigger.

Now, let's suppose that we want to find out, say, how much money we'll have after 10 years of investing with this stock. One way to find out would be to continue the above chart until 10 years have passed (that's what we did in Excel, and it's great!). Another way is to look for a pattern…using my old buddy and yours, algebra. ☺

Years Passed	Y = Value of Investment
0	$1000
1	$1000 + 0.036($1000) = $1000(1.036) = $1036
2	$1036(1.036) = $1073.30 (but, also – since $1036 is $1000(1.036), we can do this!) $1036(1.036) = $1000(1.036)(1.036) = $1000(1.036)^2 = $1073.30 (and we can keep this up! Below, I'll list the previous year's money in **green** and the current year's interest added in **red**!)
3	**$1000(1.036)^2**(**1.036**) = $1000(1.036)^3 = $1111.94
4	**$1000(1.036)^3**(**1.036**) = $1000(1.036)^4 = $1151.96

Oh my! Do you see the pattern in the exponent used and the number of years that have passed? They match![3] So, that means that after a set number of years (call it "***n***"), we'd have $1000(1.036)^n dollars in our account. Let's see if we get the same answers as we did above with our fancy new formula!

1. (**3 points**) (**w**) Let's answer the question "How much money will we have in our account after 10 years (assuming the 3.6% rate stays constant)? You can use either our new formula, or make a spreadsheet – your choice! Just show me what you used!

2. (**4 points**) (**w**) How long will it take our investment to double? Round to the nearest year. Again, show me how you got it!

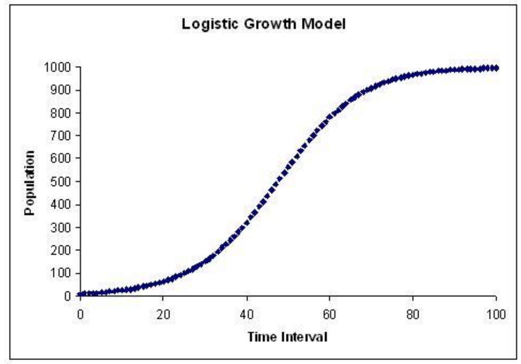
3. (**3 points**) Google the "Rule of 72" and explain how it relates to your previous answer.

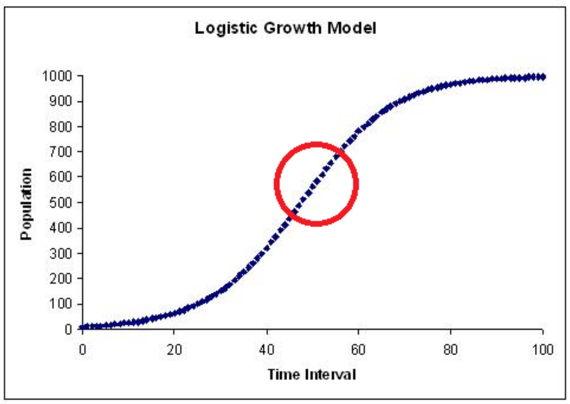
[3] Hopefully, that makes sense – the whole point of exponential data that it's formed by repeated multiplications.

Quiz 3.

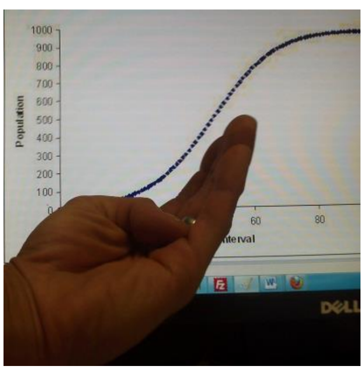
Certain things, like population growth, are well – defined by a cousin to the exponential model: the **logistic model** (we may have talked briefly about it in class). I like to think of it as a "tempered exponential growth" model: it starts exponentially, but then "slows down" as it reaches an equilibrium point.

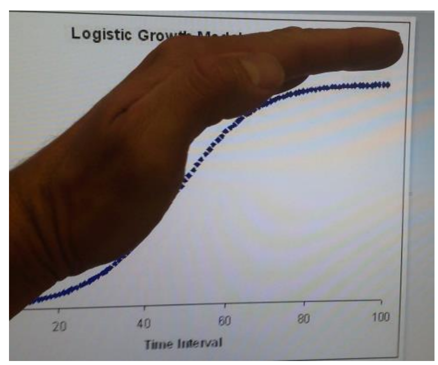
At right is a graph of a population's (logistic – looking) growth with respect to time. This graph has two "asymptotes"…that is, two places that the graph appears to "get close to, but never cross"[4]. One asymptote is the ***x*** – axis, which makes sense: the population is starting near 0, and then growing. The growth then appears to be exponential for a bit, until it slows (more on that in a minute), and then it starts to level off (in this curve, it begins to approach 1000). The second asymptote, therefore, is the line ***y***= 1000, which would be called the carry capacity of this population (the population is stabilizing towards that value).





Take a look at the graph again, at left…in particular, the area I've circled. This area of the graph is called an "inflection point"; it's the place (in this case, right around a time unit of 50) where the growth appears to begin to slow. In fancy mathematical terms, the curve goes from "concave up" to "concave down" (in less math – geek terms, if you were using your hand to show the "curve", concave up would mean your hand is facing up, and concave down means your hand was facing down):





Inflection points are one way to discern whether a logistic model is a sensible choice for your data. Let's see how to use these ideas, based on something I actually noticed when volunteering in our son Max's 4th grade classroom – the spread of a rumor!

Now, I have NO idea how 4th grade rumors start. However, they do, often, seem to start with 1 kiddo. And then that kiddo tells another kiddo. And then each of them tells someone else. However, eventually, they run out of kiddos to tell (the classroom's only so big).

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[4] The ***awesome*** word "asymptote" is actually a joining of three Greek affixes: 1) the prefix "a-" is shortened from "an-", which means "not". We use that all of the time (antonym, antithesis, antichrist, etc.); 2) the preposition "sym" comes from "sum", which means "together with". We use that when we add; 3) the "ptote" part comes from the Greek verb "piptein", which means "to fall". Putting these words together literally, an asymptote is a curve that some other curve "doesn't fall together with". In other words, the second curve 'runs alongside' its asymptote, getting infinitely closer (*paraphrased from the Math Forum*).

So, I decided to simulate the spread of this rumor was created by randomly selecting one student to know the rumor. 5 seconds later, the initial student then randomly selects one of their classmates, and shares the rumor with them. Now, 2 students know. 5 seconds later, each of ***those*** two randomly selects a student to tell (now 4 know). And so forth, and so one. Seems exponential right?

Well, if you've ever been in a 4th grade classroom, that "exponential" guess would break down pretty quickly. What inevitably happens is that someone rushes to tell someone else, but the second person goes, "Dude[5] – I already ***know*** that!" And then, 5 seconds later they try again, on some other person. Eventually, everyone in the room knows. ☺

Elapsed Time	Number of kiddos who know the rumor!
0	1
5	2
10	4
15	8
20	13
25	20
30	24
35	25
40	26
45	26
50	26
55	26
60	26

1. (**2 points**) At what time does the "perfect exponential doubling" behavior break down?

2. (**2 points**) What must have started happening at this time?

OK! Next, go ahead and visit the site https://www.desmos.com/calculator/naf1qogfjn.

On this page, you'll see a logistic model creator that you can put data into. I did; that's part of it at right. As you do this, you'll notice a graph changing next to the data; this is both a graph of the data you're entering (in **green**) and the "best fit" logistic model (in **black**). To see it well, you'll have to zoom out (I used the little roller ball on my mouse, but it appears that you can also use the "+" and "-" buttons at top right).

x_1	y_1
0	1
5	2
10	4
15	8
20	13
25	20

3. (**2 points**) Left – click that cool looking little green circle thing next to "y1". This turns off your data, so you're only looking a the smoothed model. Does your answer in #2 above agree pretty well with the graph you see? Why or why not?

4. (**2 point**) One asymptote for this model is clearly the ***x*** – axis. What is the other one? Hint: it's a horizontal line that the smoothed curve appears to "hone in " on.

5. (**2 points**) What does this second asymptote represent, in the context of this situation?

That is, if you said it's the line "y = 16", I would say "What's the '16' mean?" **Hint**; it's not y = 16. ☺

[5] Yep. They call themselves "dude".

Quiz 4.

In the chart below, we see the population milestones of the human race, courtesy of the UN. We also see some predictions that the UN has proposed as to when the human race will hit 7, 8 and 9 billion people (http://esa.un.org/wpp/Excel-Data/population.htm...the data below was crunched/predicted in the early 2000's; predictions are in **green**). This is an important data set, for it allows humanity to plan for the future with respect to housing, food, resources, etc.

World population (billions)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Year	1804	1927	1960	1974	1987	1999	**2011**	**2024**	**2042**

Using the tool from the quiz above, find a logistic model for the world population (in billions) as a function of years **after 1800**. That is, 1804 would be 4, 1927 would be 127, etc. Don't bother writing down the equation – it's far too cumbersome to actually ***deal*** with.

1. (**1 point**) As you enter the data, both the graph of that data (in **green**) and the smoothed curve (in **black**) begin to appear. Does the smoothed curve (the predicted populations) match the data (the actual populations) pretty well, or not well at all?

2. (**5 points**) Why does the logistic model make more sense in this case than, say, an exponential one? Hint: you might want to (re)read the quiz above to help you with this.

3. (**2 points**) What should the human race's population have been at the end of 2017 (according to your model), to the nearest tenth of a billion? To find the predicted value, left – click on the smoothed logistic curve and slide your mouse pointer until the ***x***- value lands on the year you need.

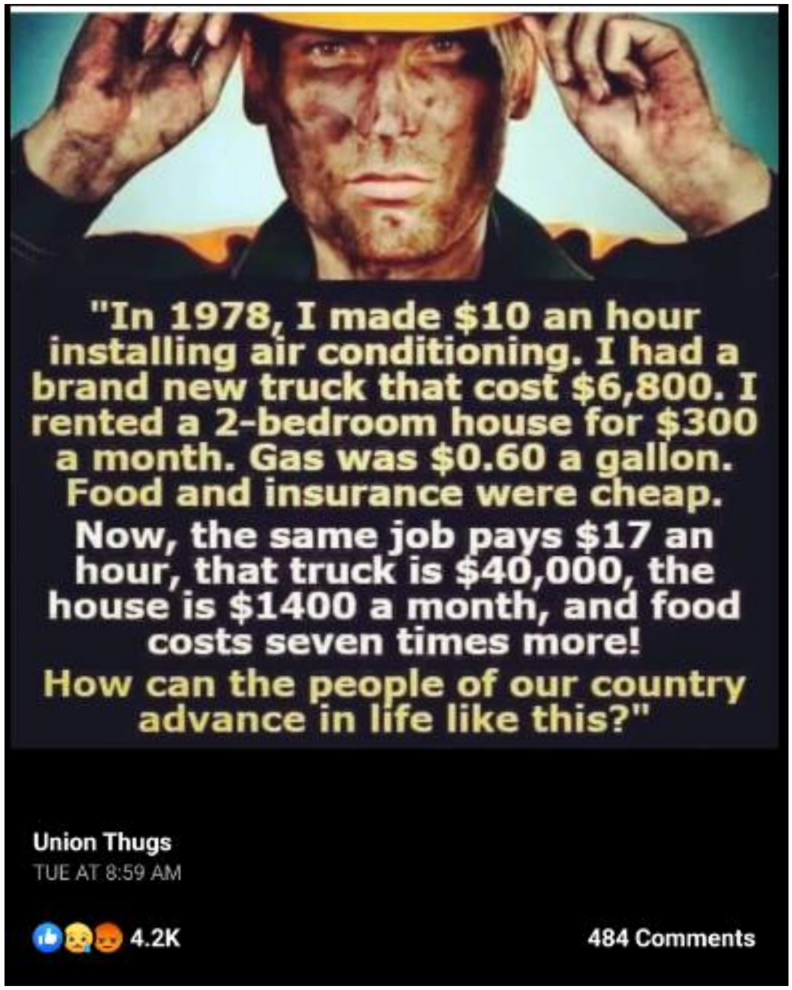
Here's a screen shot of what the world's population "was" at the end of 2017[6]:

4. (**2 points**) What is the apparent "carry capacity" of this model, to the nearest billion? That is, your model tends to "level off" at which population number?

5. (**extra 2 points**) (**w**) Using the non-predicted data, verify my claim that "the human race doubles approximately every 40 years".

[6] Notice that the predicted 7 billionth person was born in 2012. Not bad modeling that the UN did, eh?

Quiz 5.



Ah, memes…I love when they hint at math. Gotta be honest though – it kinda upsets me that this one a) didn't actually ***do*** the math, and b) offered no sourcing for their claims. Let's help them with the first part, at least!

1. (**2 points**) What message is the meme attempting to get across? Write s sentence or two.

2. (**5 points**) (**w**) Figure the average inflation rate for each of the items "hourly wage", "brand new truck", "rent" and "food" over the 40-year period implied by the meme (we'll skip "gas" and "insurance", since they didn't give us their 2018 levels). Speaking of 2018 - this meme came out in early 2019, so we should really use the last full year for annual inflation, which would be 2018. Be sure to show me how you do each one!

3. (**1 point**) What was the <u>**actual**</u> average rate of economic inflation in the US during this time period? You can Google it if you're tired of running the numbers.

4. (**2 points**) Is the meme's message correct, in your opinion? Why or why not? Write a sentence or two.

Quiz 6.

In this quiz, we're going to expand on the compounding ideas we've been discussing, and going to do one better: we're going to invest money, not once, but ***repeatedly***! I call these types of savings plans "annuities", but I'm not sure if that's really the correct term. doesn't matter – what matters is that you understand how they work. Let's watch this video to see!

1. (**1 point**) After watching that video, create a similar sheet for the second student. Take a screenshot of it and include it as your answer to this question!

2. (**1 point**) Suppose both of these students began their 25-year plans at age 20. At age 45, you saw how much was in the first student's plan. How much is in the second student's plan?

3. (**1 point**) Who earned more overall in their account after 25 years, student 1 or student 2[7]?

4. (**1 point**) How much more money did student 2 have to deposit over these 25 years than student 1? Answer as either a multiple or a percent (for example, if student 1 deposited a total of $5000 and student 2 deposited a total of $7500, then student 2 deposited 50% more, or 1.5 times as much).

5. (**1 point**) How much more interest did student 1 earn than student 2? Answer the same way you did in the last one.

6. (**5 points**) Summarize the lesson here in a couple of sentences. And it's more than just "saving money in bank accounts is good." ☺

[7] There are definitely taxes that you'd have to pay on any interest-bearing account, for sure. But assuming these students are in roughly the same tax bracket, they'd be roughly the same.