7.2 Exponential Patterns

- Recognize and describe an exponential pattern.
- Use an exponential pattern to predict a future event.
- Compare exponential and logistic growth.

Study Tip

Notice the difference between linear and exponential patterns. With linear patterns, successive numbers increase or decrease by the same *amount*. With exponential patterns, successive numbers increase or decrease by the same *percent*.

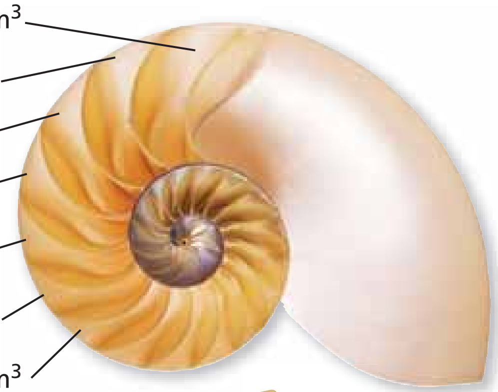
Recognizing an Exponential Pattern

A sequence of numbers has an **exponential pattern** when each successive number increases (or decreases) by the same percent. Here are some examples of exponential patterns you have already studied in this text.

- Growth of a bacteria culture (Example 1, page 152)
- Growth of a mouse population during a mouse plague (Example 3, page 154)
- Decrease in the atmospheric pressure with increasing height (Example 2, page 175)
- Decrease in the amount of a drug in your bloodstream (Example 3, page 176)

EXAMPLE 1 Recognizing an Exponential Pattern

Describe the pattern for the volumes of consecutive chambers in the shell of a chambered nautilus.



SOLUTION

It helps to organize the data in a table.

DATA

Chamber	1	2	3	4	5	6	7
Volume (cm^3)	0.836	0.889	0.945	1.005	1.068	1.135	1.207

Begin by checking the differences of consecutive volumes to conclude that the pattern is *not linear*. Then find the *ratios* of consecutive volumes.

$$\frac{0.889}{0.836} \approx 1.063 \qquad \frac{0.945}{0.889} \approx 1.063 \qquad \frac{1.005}{0.945} \approx 1.063$$

$$\frac{1.068}{1.005} \approx 1.063 \qquad \frac{1.135}{1.068} \approx 1.063 \qquad \frac{1.207}{1.135} \approx 1.063$$

The volume of each chamber is about 6.3% greater than the volume of the previous chamber. So, the pattern is exponential.

The exponential growth pattern of the chambers in a chambered nautilus was first recorded by the French philosopher René Descartes in 1638.

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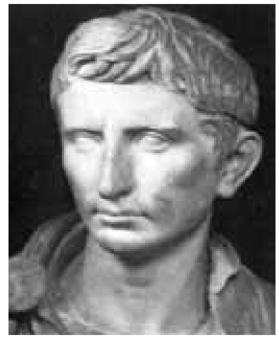
Use a spreadsheet to extend the pattern in Example 1 to 24 chambers. Then make a scatter plot of the data and describe the graph.

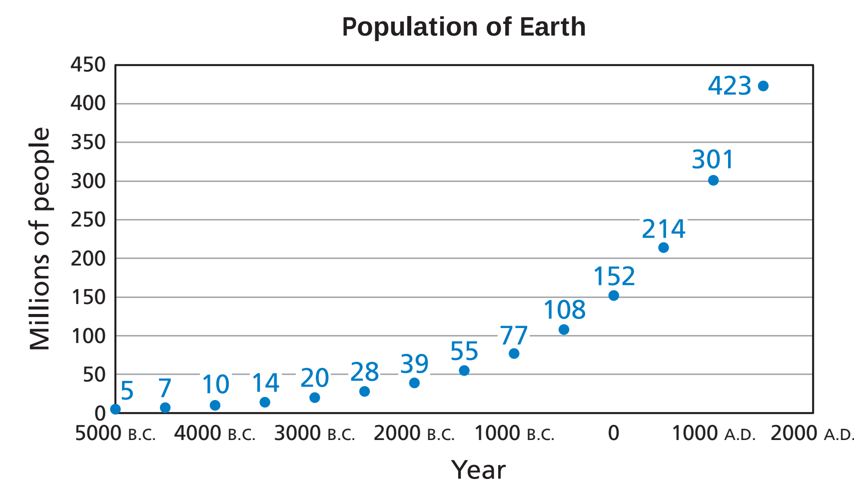
EXAMPLE 2 Describing an Exponential Pattern

From 5000 B.C. through 1500 A.D., the population of Earth followed a growth pattern that was roughly exponential. Describe the growth pattern in words.



1 B.C. Augustus Caesar controlled most of the Mediterranean world.

1000 A.D. The Song Dynasty in China had about one-fifth of the world's population.



SOLUTION

Begin by finding the ratios of consecutive populations.

$\frac{7}{5} = 1.40$ $\frac{10}{7} \approx 1.43$ $\frac{14}{10} = 1.40$ $\frac{20}{14} \approx 1.43$ $\frac{28}{20} = 1.40$

$\frac{39}{28} \approx 1.39$ $\frac{55}{39} \approx 1.41$ $\frac{77}{55} = 1.40$ $\frac{108}{77} \approx 1.40$ $\frac{152}{108} \approx 1.41$

$\frac{214}{152} \approx 1.41$ $\frac{301}{214} \approx 1.41$ $\frac{423}{301} \approx 1.41$

From these Earth population estimates, you can say that Earth's population was increasing by about 40% every 500 years.

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Did the growth pattern described in Example 2 continue through the next 500 years, up through the year 2000? If not, why didn't the pattern continue?

1000 B.C. Approximate beginning of the Iron Age

2000 B.C. Beginning of the Middle Kingdom in Egypt

3000 B.C. Stonehenge is built in England.

4000 B.C. Civilization begins to develop in Mesopotamia.

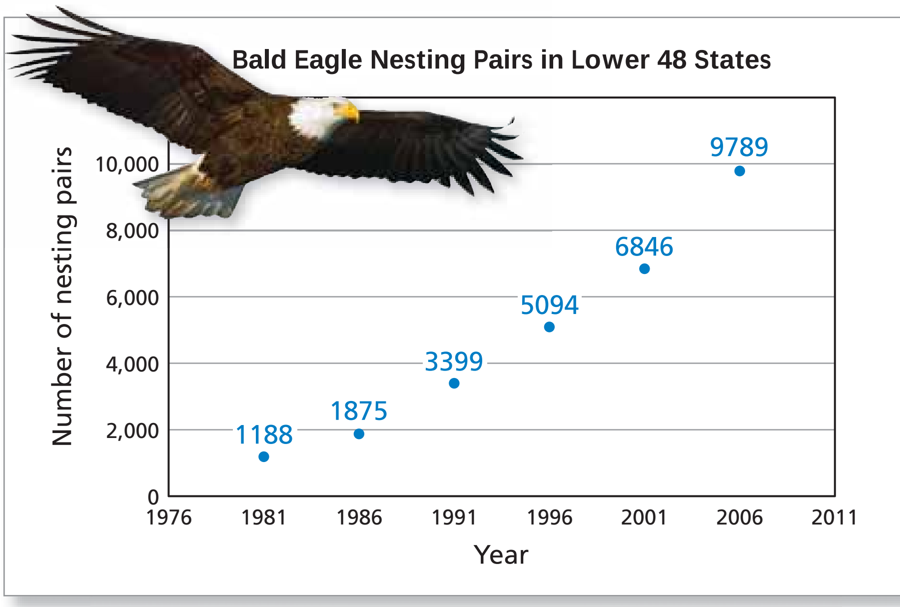
The mission of the U.S. Fish and Wildlife Service is "to work with others to conserve, protect, and enhance fish, wildlife, and plants and their habitats for the continuing benefit of the American people."

Using an Exponential Pattern to Predict a Future Event

EXAMPLE 3 Predicting a Future Event

It is estimated that in 1782 there were about 100,000 nesting bald eagles in the United States. By the 1960s, this number had dropped to about 500 nesting pairs. This decline was attributed to loss of habitat, loss of prey, hunting, and the use of the pesticide DDT.

The 1940 Bald Eagle Protection Act prohibited the trapping and killing of the birds. In 1967, the bald eagle was declared an endangered species in the United States. With protection, the nesting pair population began to increase, as shown in the graph. Finally, in 2007, the bald eagle was removed from the list of endangered and threatened species.



Assume that this recovery pattern continued. Estimate the number of nesting pairs of bald eagles in the lower 48 states in 2011.

SOLUTION

Begin by finding the ratios of consecutive populations.

$$\frac{1875}{1188} \approx 1.58 \qquad \frac{3399}{1875} \approx 1.81 \qquad \frac{5094}{3399} \approx 1.50$$

$$\frac{6846}{5094} \approx 1.34 \qquad \frac{9789}{6846} \approx 1.43$$

From the data, it appears that the population increased by about 50% every 5 years. So, from 2006 to 2011, you can estimate that the population increased to 1.5(9789), or about 14,700 nesting pairs.

Study Tip

Using a computer and an exponential regression program, you can find that the best estimate for the increase (every 5 years) for the data in Example 3 is 52.8%.

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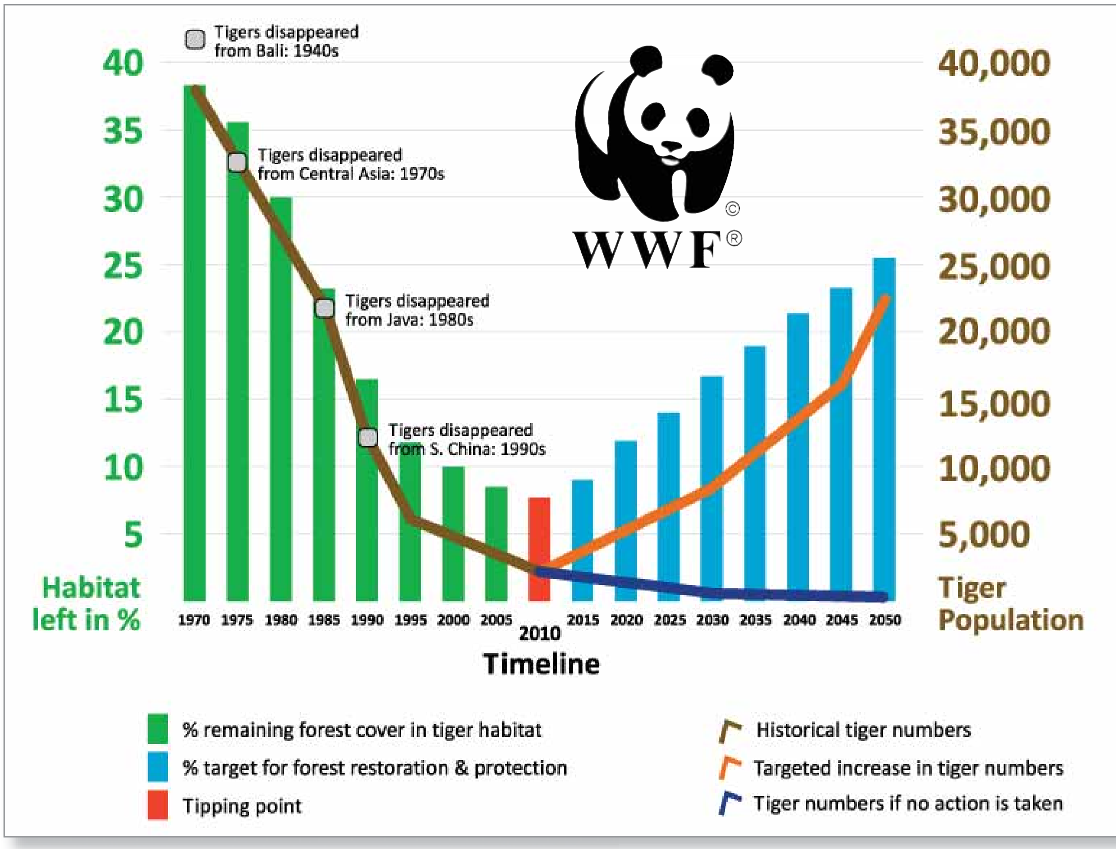
Suppose the recovery pattern continued for another 5 years. Predict the number of nesting pairs in 2016.

EXAMPLE 4 Predicting a Future Event

Discuss the following graph prepared by the World Wildlife Fund. What exponential pattern can you see in the graph?



SOLUTION

From the graph, the estimated tiger population appears to be decreasing with an exponential pattern, as follows.

1985	1990	1995	2000	2005
22,000	12,500	6000	5000	3800

$$\frac{12{,}500}{22{,}000} \approx 0.568 \qquad \frac{6000}{12{,}500} = 0.480 \qquad \frac{5000}{6000} \approx 0.833 \qquad \frac{3800}{5000} = 0.760$$

Although the rate of decrease in each 5-year period varies, you need to remember that these data are difficult to collect and consequently are only an approximation. Even so, it appears that the tiger population is decreasing by almost 70% every 5 years.

Checkpoint

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Estimate the percent of remaining tiger habitat from 1985 through 2010. Describe the pattern.

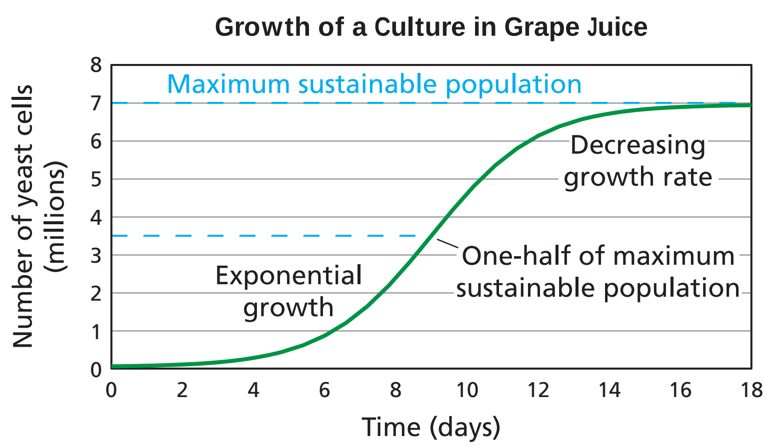
Comparing Exponential and Logistic Growth

Exponential growth can only occur for a limited time in nature. Eventually, the quantity that is growing reaches physical boundaries. The resulting growth is called **logistic growth**.

Yeasts are single-celled organisms. Most reproduce by asexual budding (splitting to form two new yeast cells). When yeast cells lack oxygen, they die and produce alcohol. This process is called fermentation.

EXAMPLE 5 Comparing Exponential and Logistic Growth

The graph shows the growth of a culture of yeast cells that is introduced into a container of grape juice. Describe the growth.



SOLUTION

During the exponential growth stage, most of the energy of the yeast culture is devoted to reproducing itself. To do this, it uses the natural sugar that is in the grape juice. Wine fermentation has two stages called aerobic (with oxygen) and anaerobic (without oxygen) fermentations. After a few days in the first stage, most of the sugar and other nutrients in the grape juice are depleted. At this point, the oxygen source is removed and the growth rate of the yeast starts to decrease. Eventually, the yeast cells die (this is not shown in the graph). So, the population is limited by the food and oxygen available.

Thomas Malthus is known for his theories on population growth. He claimed that populations are checked by famine, disease, and widespread mortality.

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What is your opinion about the sustainable population level of humans on Earth? Do you agree with Thomas Malthus, who predicted that the human population will grow exponentially, creating a permanent class of poor? Explain your reasoning.

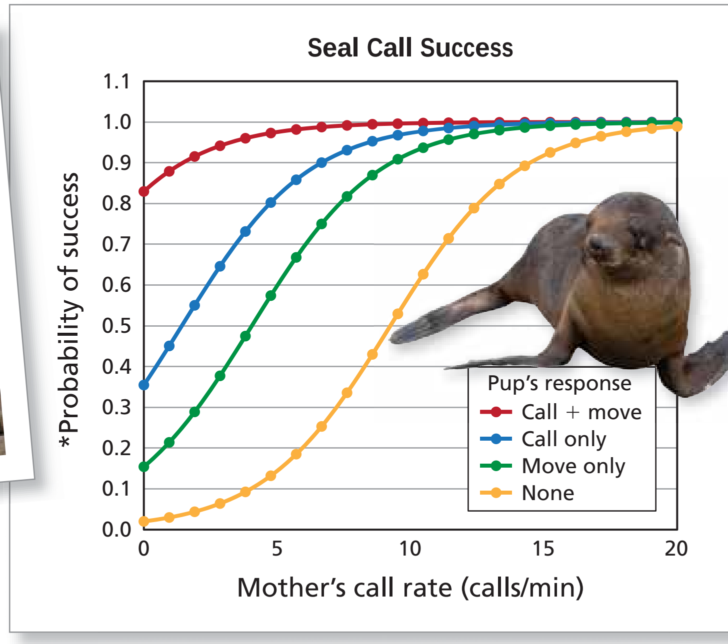
"**1.** Population is necessarily limited by the means of subsistence.

2. Population invariably increases where the means of subsistence increase, unless prevented by some very powerful and obvious checks.

3. These checks, and the checks which repress the superior, power of population, and keep its effects on a level with the means of subsistence, are all resolvable into moral restraint, vice and misery." *Thomas Malthus*

EXAMPLE 6 Comparing Logistic Patterns

The graph shows four different strategies used by mother and baby fur seals to locate each other after the mother returns from hunting. Discuss the strategies. Which is more effective?



After a fur seal pup is born, its mother nurses it for about 4 months. During this time, the mother makes frequent trips to sea to forage for food. Each time a mother returns from hunting, she has to locate her pup in the colony, which can have thousands of baby and adult seals.

Probability is discussed in Chapter 8.

SOLUTION

Here are some general observations about the graph.

1. In each strategy, the mother has more success locating her pup when she increases her bark rate per minute.
2. As the bark rate increases, the probability of success increases logistically.
3. If the mother calls at a rate of 20 barks per minute, she is almost certain to locate her pup, regardless of the pup's response.

Here are some observations relative to the pup's response.

- **The pup calls and moves.** This is the best strategy for the pup. By calling and moving, there is a good chance that its mother will find it.
- **The pup only calls.** This is the second-best strategy for the pup. By calling, there is still a good chance that its mother can hear it through the noise of the colony.
- **The pup only moves.** This is not a good strategy for the pup. If its mother is calling at the rate of only 5 barks per minute, there is only a 60% chance that its mother will find it.
- **The pup does nothing.** This is the worst strategy for the pup. A pup who is too weak to call or move does not have a good chance of being found.

Checkpoint

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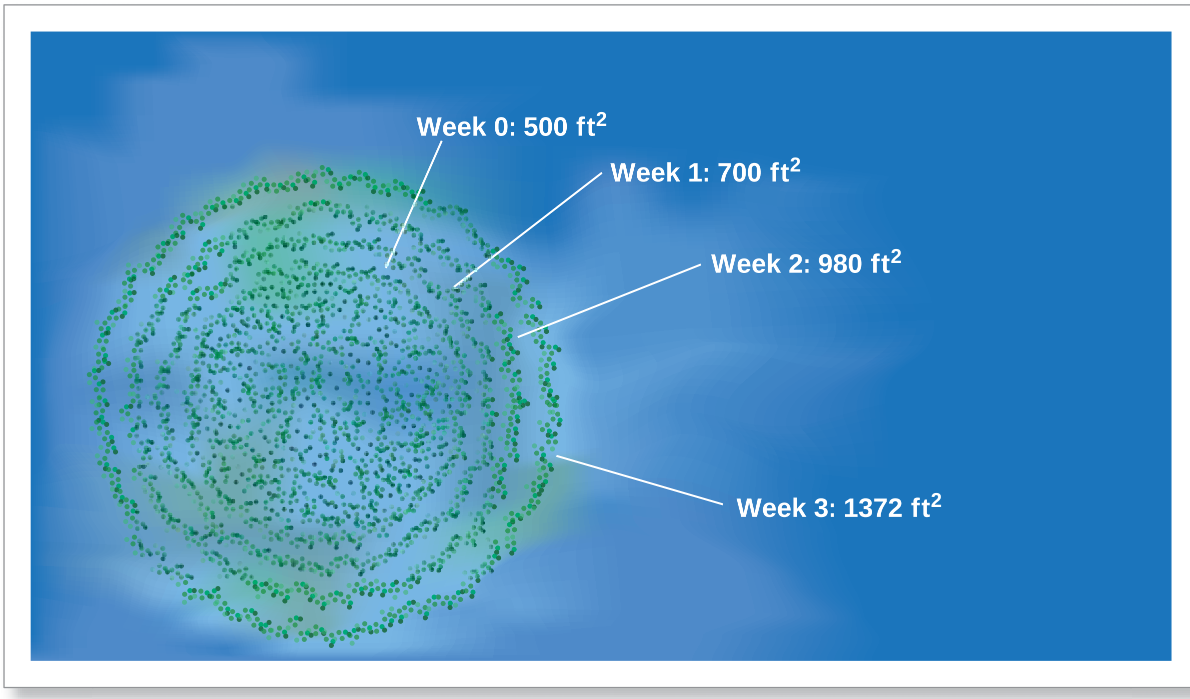
The above graph applies each time the mother goes to sea for food. Explain how the pup's chance for survival changes with multiple trips by the mother.

7.2 Exercises

DATA **Water Hyacinth** **An invasive species of water hyacinth is spreading over the surface of a lake. The figure shows the surface area covered by the water hyacinth over a 3-week period. In Exercises 1–4, use the figure.** ***(See Examples 1 and 2.)***



1. Is the pattern linear? Explain your reasoning.

2. At what rate is the surface area covered by the water hyacinth increasing?

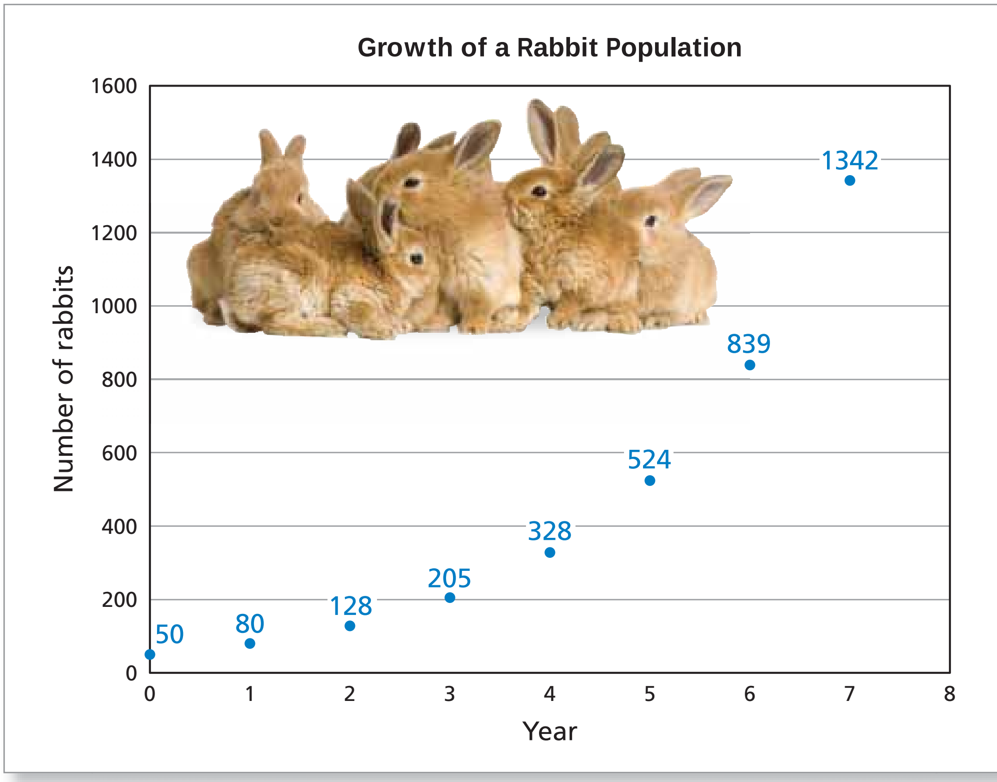
3. Use a spreadsheet to extend the pattern to 20 weeks. Then make a scatter plot of the data and describe the graph.

4. The surface area of the lake is about 800,000 square feet. How many weeks does it take the water hyacinth to cover the entire lake?

DATA **5.** **Invasive Species** An invasive species of water plant covers 1500 square feet of the surface of a lake. The lake has a surface area of about 2,500,000 square feet. The surface area covered by the plant increases by 60% each week. Make a table and a scatter plot showing the surface area covered by the plant until the plant covers the entire lake. *(See Examples 1 and 2.)*

DATA **6.** **Invasive Species** Suppose in Exercise 5 that the surface area covered by the plant increases by only 20% each week. How much longer does it take the plant to cover the entire lake? *(See Examples 1 and 2.)*

Rabbits **A rabbit population is introduced to a new area. The graph shows the growth of the rabbit population. In Exercises 7–12, use the graph.** *(See Example 3.)*



7. What does the population for year 0 represent?

8. At what rate is the rabbit population increasing?

9. Suppose the population growth continued for another year. Predict the number of rabbits in year 8.

10. Suppose the population growth continued for another 3 years. Predict the number of rabbits in year 10.

11. When does the rabbit population exceed 3000?

12. When does the rabbit population exceed 6000?

DATA **13.** **Population Growth** A rabbit population grows exponentially over a 10-year period. The population in year 3 is 150. The population in year 4 is 204. Predict the number of rabbits in year 10. *(See Example 3.)*

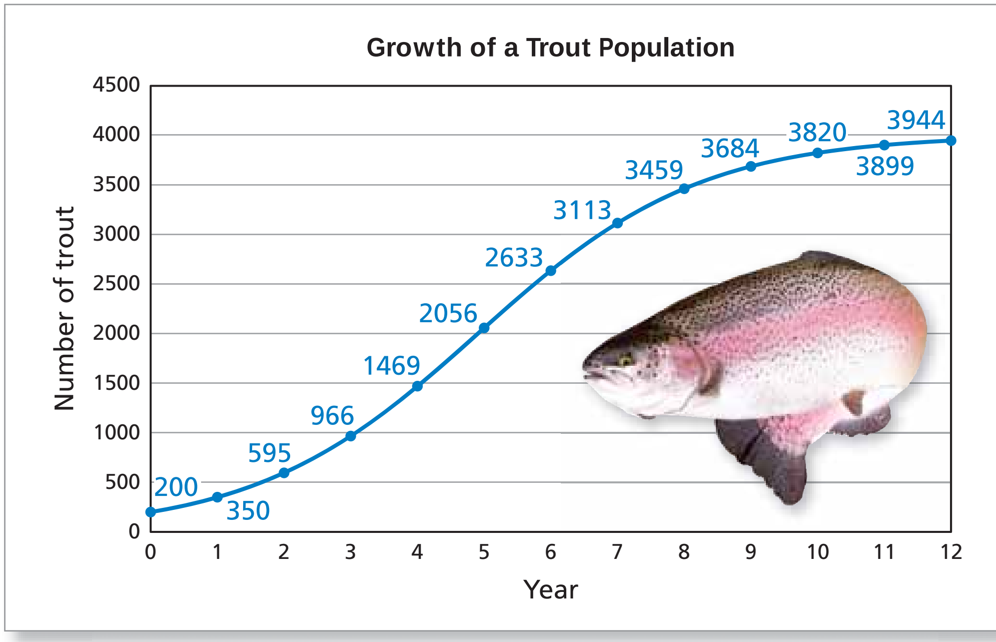
DATA **14.** **Disease Outbreak** The outbreak of a disease causes a rabbit population to decrease exponentially over a 6-year period. The population in year 2 is 1200. The population in year 3 is 960. Predict the number of rabbits in year 6. *(See Example 4.)*

Trout A lake is stocked with 200 trout. The graph shows the growth of the trout population. In Exercises 15–18, use the graph. ***(See Example 5.)***

15. What is the maximum sustainable population? Explain your reasoning.

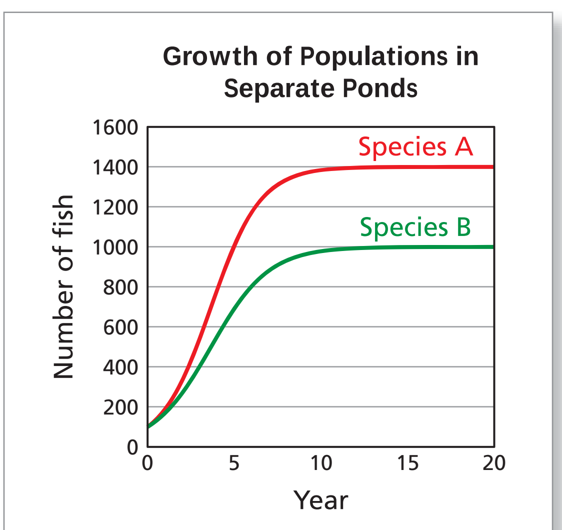
16. Make a table that shows the change in the number of trout for each year. Discuss any trends.

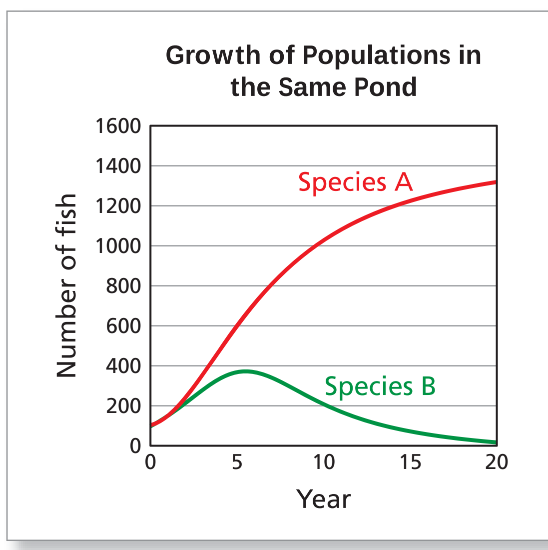
DATA **17.** Make a table that shows the percent change in the number of trout for each year. Discuss any trends.



18. Make a table comparing the number of trout for each year in the graph to the number of trout each year if the trout population grew exponentially by 60% each year. Why is exponential growth unrealistic in this situation?

Competing Species The graphs show the growth of the populations of two competing species of fish when they are released into separate ponds and when they are released into the same pond. Assume all the ponds are the same size and have the same resources. In Exercises 19 and 20, use the graphs. ***(See Example 6.)***





19. Compare the growth of the populations of the two species when they are released into separate ponds.

20. Compare the growth of the populations of the two species when they are released into the same pond.

Extending Concepts

Logistic Growth Rate **In Exercises 21–24, use the information below.**

The formula for the logistic growth rate is

$$r = r_0 \times 1 - \left(\frac{\text{population size}}{\text{maximum sustainable population}}\right)$$

where r_0 is the intrinsic growth rate in decimal form.

21. The population of squirrels in a forest is growing logistically. The intrinsic growth rate is 40% per year, and the maximum sustainable population is 5000. Find the rate at which the population is growing when the population reaches (a) 100, (b) 2000, and (c) 4500.

22. The population of raccoons in a forest is growing logistically. The intrinsic growth rate is 50% per year, and the maximum sustainable population is 2000. Find the rate at which the population is growing when the population reaches (a) 200, (b) 1000, and (c) 1800.

23. What is the growth rate when a population is at the maximum sustainable population? Explain.

24. Describe the growth rate when a population exceeds the maximum sustainable population. Explain what this represents in nature.

DATA **Superexponential Growth** **In Exercises 25 and 26, use the information below.**

A population undergoes *superexponential growth* when the growth rate increases exponentially over time.

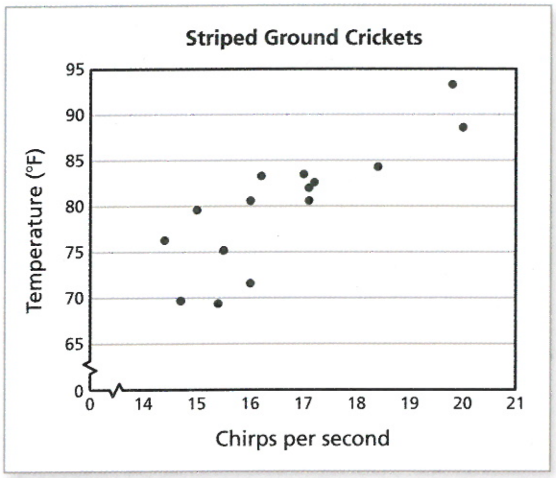
25. A population of 100 locusts is introduced to a new area. The initial growth rate is 50% per year. Make a table comparing the population of the locusts over a 10-year period when the growth rate remains constant (exponential) and when the growth rate increases by 20% each year (superexponential). Then make a scatter plot comparing the two data sets.

26. A population of 50 frogs is introduced to a new area. The initial growth rate is 60% per year. Make a table comparing the population of the frogs over a 10-year period when the growth rate remains constant (exponential) and when the growth rate increases by 10% each year (superexponential). Then make a scatter plot comparing the two data sets.

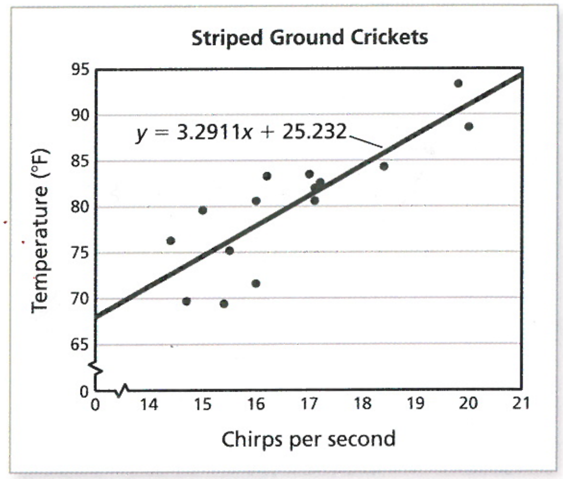
DATA **27.** **Riddle of the Lily Pad** A single lily pad lies on the surface of a pond. Each day the number of lily pads doubles until the entire pond is covered on day 30. On what day is the pond half-covered?

28. **Mutant Plant** A mutant strain of water plant covers 100 square feet of the surface of a lake. The lake has a surface area of about 1,000,000 square feet. At the end of each day, the surface area covered by the plant is double what it was at the beginning of the day minus the amount of plant cover that you clear. You can clear 5000 square feet of plant cover in 1 day. On what day do you have to begin clearing the plant cover to stop the plant from spreading across the entire lake?

19. **a.**



b.



c. About 87.8°F

d. About 97.6°F

Section 7.2 *(page 318)*

1. No; The difference between the surface area values for consecutive weeks is not constant.

3.

	A	B	C	D
1	Week	Surface area (square feet)		
2	0	500		
3	1	700		
4	2	980		
5	3	1,372		
6	4	1,921		
7	5	2,689		
8	6	3,765		
9	7	5,271		
10	8	7,379		
11	9	10,331		
12	10	14,463		
13	11	20,248		
14	12	28,347		
15	13	39,686		
16	14	55,560		
17	15	77,784		
18	16	108,898		
19	17	152,457		
20	18	213,439		
21	19	298,815		
22	20	418,341		
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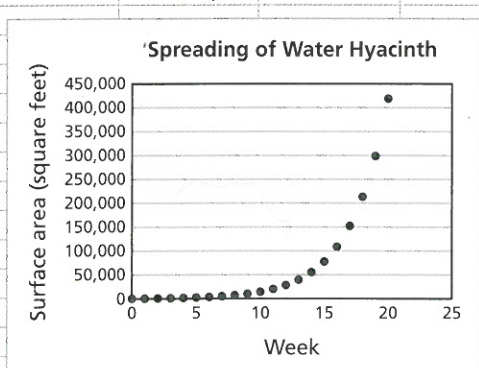
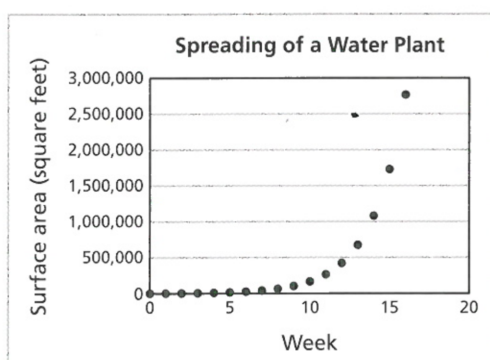


The graph shows exponential growth.

5.

Week	Surface area (square feet)
0	1,500
1	2,400
2	3,840
3	6,144
4	9,830
5	15,729
6	25,166
7	40,265
8	64,425
9	103,079
10	164,927
11	263,883
12	422,212
13	675,540
14	1,080,864
15	1,729,382
16	2,500,000



7. The initial population **9.** 2147 rabbits **11.** During year 9 **13.** 1291 rabbits

5. About 4000 trout; The population approaches but does not exceed 4000 over time.

7.

Year	Percent change in the number of trout
1	75.0
2	70.0
3	62.4
4	52.1
5	40.0
6	28.1
7	18.2
8	11.1
9	6.5
10	3.7
11	2.1
12	1.2

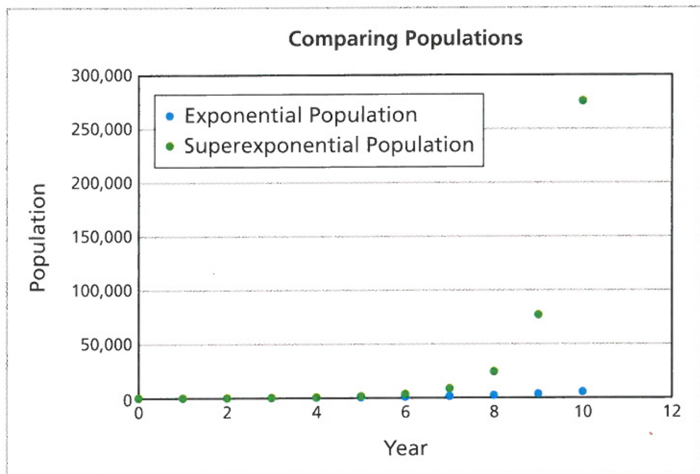
The percent of increase is decreasing each year.

19. Both species of fish experience a logistic growth pattern. Species A levels off with a population of about 1400 fish. Species B levels off with a population of about 1000 fish.

21. **a.** 39.2% **b.** 24% **c.** 4%

23. 0%; When the population reaches its maximum sustainable population, $r = r_0 \times (1 - 1) = r_0 \times 0 = 0$. This is because when a population is at its maximum, it cannot grow any larger.

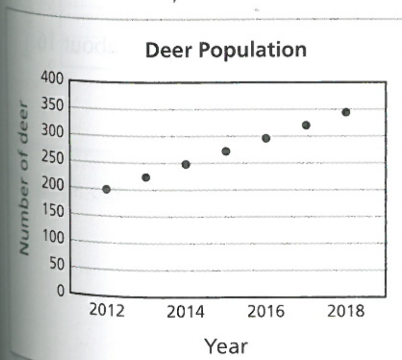
Year	Exponential population	Superexponential population
0	100	100
1	150	150
2	225	240
3	338	413
4	506	769
5	759	1567
6	1139	3517
7	1709	8768
8	2563	24,477
9	3844	77,101
10	5767	276,012



Day 29

-7.2 Quiz *(page 322)*

The pattern for Set A is linear. The deer population is projected to grow at a rate of 24 additional deer each year.



3. The pattern for Set B is exponential. The deer population is projected to grow at a rate of about 12% each year.