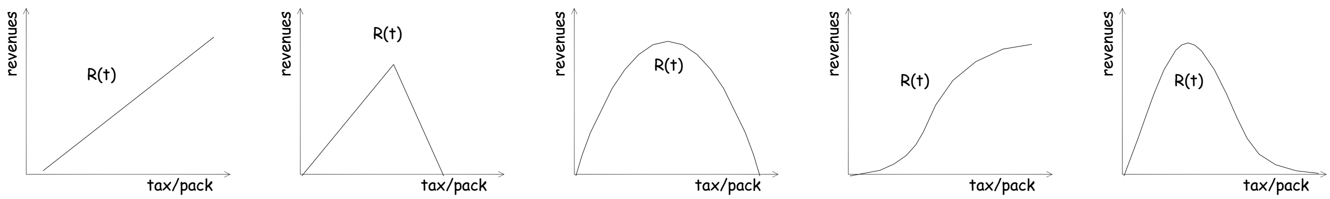
Building Mathematical Models *Franz Helfenstein* Name

Introduction

In November, 2007 Oregon voters defeated **Measure 50** which would have imposed higher state cigarette taxes to fund child health care. To better understand such *sin taxes* from a mathematical perspective, we want to model revenues as a function of cigarette taxes.



Before we begin, what does your intuition tell you? Should revenues depend on the amount of tax? Assuming revenues depend on taxes, what would you expect the general behavior of such a function to be? Pick one of the above behaviors and explain why you feel that is the most likely relation.

Building a Mathematical Model

Building mathematical models is part art and part procedure. However, there are some steps that are common to all model building.

1. Decide which variables are pertinent in the analysis.
2. Decide which variables are dependent and which are independent.
3. Take variable pairs (1 dependent and 1 independent) and find functions that describe the relationship between the variables. This can be done in one of two ways.
   (a) Use data and regression to find the function.
   (b) Use logic and algebra to find the function.
4. Once we have a set of functions we can put them together to obtain a complete model.
5. We can also use additional data to test (verify) the model.
6. Once we are satisfied the model is valid we can use the model to answer pertinent questions.

1) Deciding what variables are pertinent to the analysis.

Although there are many choices we might want to consider, we shall begin with the following variables: year, percentage of smokers, total number of smokers, tax/pack, cost of cigarettes/pack without tax (COGS), total cost of cigarettes/pack (net cost), tax revenues.

2) Deciding which variable is dependent and which is independent.

The year along with the corresponding tax/pack is shown in the table. *Which variable is dependent, which is independent?*

Year	2000	2001	2002	2003	2004
Tax	$0.29	$0.32	$0.35	$0.43	$0.57

Tax depends on Year

3) Finding a function that describes the relationship Tax vs. Year *i.e. Tax is a function of year.*

(a) Enter the independent variable into L_1 and the dependent variable into L_2. (Use year = 0 for 2000)
(b) Set the Window. I suggest $[0, 7] \times [0, 5]$
(c) Plot the data.
(d) Use regression to find an appropriate linear model for tax as a function of year. i.e. tax(year)

Y_1 = tax(year) = 0.067 X + 0.258 *keep 3 digits and store your result in Y_1*

4) Between year and cigarette cost, which variable is dependent and which is independent?

Cost depends on Year

5) Finding a function that describes the relationship between Year and Cigarette Cost

In 2000, the average retail cost of goods (without state taxes) was \$2.88. Each year the retail cost typically has increased by 3%. Thus retail costs have increased as an exponential function much like a loan balance. Use this information to create the retail cost of goods (COGS) as a function of year i.e COGS(yr). Use year = 0 for 2000

Y_2 = COGS(yr) = 2.88×1.03^X *store your result in Y_2*

6) Finding the total cost of cigarettes as a function of time i.e. Cost(yr). COGS + State tax = Total Cost

$Y_1 + Y_2$ = Cost(yr) = $0.067X + 0.258 + 2.88 \times 1.03^X$ *store your result in Y_3*

7) Use your TI TABLE to fill in this table.

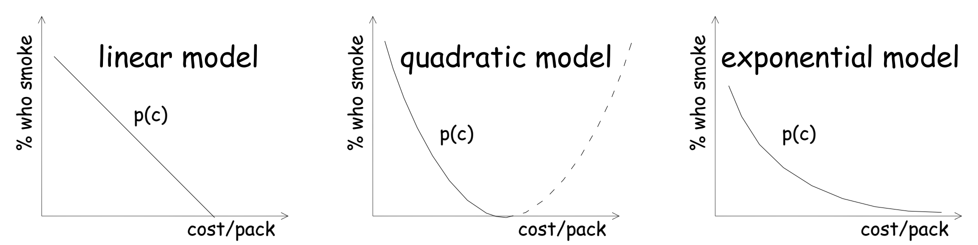
Use TBLSET, Tblstart = 0 and ΔTbl = 1.

Do not use the current tax! Be sure to use your function values.

Year (X)	2000	2001	2002	2003	2004	2007
Tax (Y_1)	\$0.26	\$0.33	\$0.39	\$0.46	\$0.53	\$0.73
COGS (Y_2)	\$2.88	\$2.97	\$3.06	\$3.15	\$3.34	\$3.54
Cost (Y_3)	3.14	\$3.29	\$3.45	\$3.61	\$3.77	\$4.27
% Smokers	44.8%	42.2%	40.8%	37.8%	35.4%	34.9%

8) Finding a function that describes the relationship between Smokers and Cost/Pack

Although smokers may not actually quit but just smoke less it still seems reasonable the percentage of smokers should depend on the cost of cigarettes. But which model is the most logical choice?



Explain!

9) Find the function %Smokers as a function of cost.

(a) From the table in (7), enter the Cost into L_3 and Percentage of Smokers into L_4.
(b) Set the Window. I suggest $[0, 10] \times [0, 1]$
(c) Plot L_4 vs. L_3.
(d) Run regression to find the model for %-Smokers as a function of cost i.e. %Smokers(cost).

Y_4 = %Smokers(cost) = 0.8947×0.7943^X *keep at least 4 digits and store your result in Y_4*

10) Use Y_4 to predict the percent of smokers if cigarettes were free. $Y_4(0) = 89\%$

Does this result seem reasonable? Explain!

11) Use Y_4 to predict the total cost that would drop the percent of smokers to 10%.

Solve $Y_4 = 10\%$. Cost = \$9.52

12) Compute the State tax that would be necessary to drop the smoking rate to 10%.

Tax = Cost − 2007-COGS   Lookup the 2007-COGS value from the table you made on page 2.

State Tax necessary to drop smoking to 10% of Oregonians = \$9.52 − \$3.54 = \$5.98

In the 2000 Census there were 3,421,399 people living in Oregon. Oregon's population growth has averaged 1.37% since 2000. This growth is exponential much like a loan balance. Let P(t) be the number of people in Oregon by year starting with t = 0 in 2000.

13) Create the population of Oregon as a function of time.

$P(t) = 3{,}421{,}399 \times 1.0137^t$ *store your result in Y_5*

14) Use P(t) to find the population in 2007. Total Population in 2007 = P(7) = 3,763,309

15) Use your Y_4 function to predict the total number of smokers in 2007.

Smokers in 2007 = 0.349 × P(7) = 0.349 × 3,763,309 = 1,313,395 smokers

16) Compute the cost under Measure 50. Measure 50 proposes a State tax increase of \$0.845 per pack. Lookup the 2007-Cost value from the table you made on page 2. New Cost = Old Cost + 0.845.

New Cost = \$4.27 + \$0.845 = \$5.115

17) Use your Y_4 function to predict the total number of smokers under Measure 50.

Store New Cost in X. Then execute Y_4 to get the %-Smokers under Measure 50 tax.

%-Smokers under Measure 50 = 0.2755 = 27.55%

Smokers under Measure 50 = %-Smokers × P(7) = 0.2755 × 3,763,309 = 1,036,736 smokers

18) If Measure 50 goes into effect, how many people would quit smoking based on our model so far?

Quit Smoking = 1,313,395 − 1,036,736 = 276,659

Revenue from Cigarette Tax

To maximize the tax benefits of a sin tax we must consider revenues as a function of tax. As taxes increase, the cost/pack increases so the number of smokers declines. In addition, the number of packs a smoker buys will likely decline as well. At first the State will collect more revenue but as taxes rise, eventually no one will be able to afford to smoke and the revenues will decline. We have the revenue as a function of State cigarette taxes as

19) Compute the revenues as a function of tax.

Revenues(tax) = (population) × (%-smokers) × (avg packs/day) × (365 days/yr) × (state tax)

Assume for the moment that smokers continue to buy about 1 pack/day.

Revenues(tax) = (2007 population) × Y_4(new cost) × (1) × (365) × (tax)

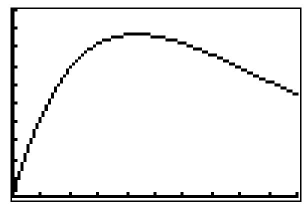
= P(7) × Y_4(2007-COGS + tax) × 365 × tax *tax is the independent variable*

Y_5 = Revenue(tax) = 3,763,309 × 0.8947 × $0.7943^{(3.54 + X)}$ × 365 X *store your result in* Y_5

20) Graph Y_5 and determine the tax that yields the most revenue.

I suggest a window of [0, 10] × [0, 10^9].

If you have done everything correctly, you should get the graph shown here.



optimum tax = $4.34 max revenue = $868,777,016

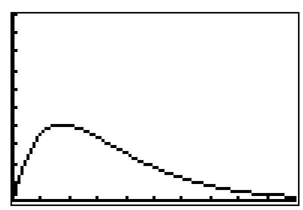
BONUS

A more realistic assumption is that some smokers would continue smoking but just smoke less. Suppose the amount of smoking (packs/day) is a function of cost. Let B(c) denote packs bought per day as a function of cost. Use the following information to find B(c).

B(c) = 4 × 0.7071^X	Total Cost	$4.00	$6.00	$8.00	$10.00
	packs/day	1	½	¼	⅛

Y_6 = Revenue(tax) = Y_5 × B(tax + 2007-COGS) = Y_5 × B(X + 3.54) = Y_5 × 4 × $0.7071^{(3.54 + X)}$

Graph Y_6 and compute the optimum tax and max revenue.



optimum tax = $1.73 max revenue = $406,752,100